PARAMETER ESTIMATION OF STRUCTURES FROM STATIC STRAIN MEasureMENTS. I: FORMULATION

By Masoud Sanayei,1 Member, ASCE, and Michael J. Saletnik,2 Associate Member, ASCE

ABSTRACT: A method for structural parameter identification utilizing elemental strain measurements is presented. By using a subset of applied static forces and a subset of measured strains, element stiffness parameters of all or a portion of a finite element model of the structure are identified. This method preserves the structural connectivity and determines changes in cross-sectional properties at the element level, including large changes or element failures for stable structures. Strain measurements do not require a frame of reference; this makes them superior to static displacement measurements. Strain measurements are also more accurate than ordinary displacement measurements and can easily be used on bridges, buildings, and space structures. The identified cross-sectional element properties can be used for damage assessment and load rating of structures. Two numerical examples, including two-dimensional (2D) truss and frame structures, are presented and the element stiffnesses are successfully and accurately evaluated.

INTRODUCTION

Structural parameter estimation is a powerful tool capable of updating an analytical model using a set of measured test data. An optimization-based method is proposed to adjust the parameters of the finite element model (FEM), minimizing the difference between the analytical response and nondestructive test (NDT) data. Parameters of the FEM can be any appropriate geometric or material property, including cross-sectional properties of the structural components. Parameter adjustments based on trial and error fail for most structures due to the large number of possible parameter values. Using parameter estimation results, major differences between the estimated and expected parameters are classified as damage. For parameter estimation the applied forces are chosen so that structural responses are linear-elastic despite some degree of damage that structural components may have experienced.

The static finite element method by design uses displacement measurements from applied loads. Measuring displacements on full-scale structures, however, can be a difficult task; a frame of reference must be established. For contact measurements, this can entail the construction of a secondary structure on an independent foundation. This can be costly and difficult. Although noncontact displacement measurement techniques are available, such techniques can be costly and difficult as well. Strain gauges, however, do not face these limitations. In measuring static strains using strain gauges, there are several sources of small errors (e.g., temperature effects, nonlinearities, balancing the bridge, and electronic noise) to deal with and control. If this is done correctly, it is possible to make static strain measurements with a higher precision than static displacement measurements using dial gauges. Although the surface preparation steps of gauge application can be lengthy and complex, once the gauges are installed all further steps, such as electrical zeroing and measurement reading are controlled by the strain gauge meter. Strains on a structural surface are caused by both bending and axial deformations of the member; therefore, strain measurements can capture element behavior well. Although strain gauges are like any other measuring device, not perfect, there are sufficient advantages to merit their use in many situations particularly for large structures whose displacements would require considerable labor to measure. Static strain measurements are definitely more involved than displacement measurements; however, they are not as complicated as strain or acceleration measurements in vibration testing. These factors make static strain measurements more attractive than static displacement measurements for NDT. For these reasons, this paper presents a new technique for using static strain measurements for parameter estimation. The companion paper, Sanayei and Saletnik (1996), studies the impact of measurement noise on the parameter estimates and presents a heuristic method for the design and setup of nondestructive tests.

Most previous work using NDT data has been done with dynamic, not static, excitation. Although there are a very large number of quality papers published in this area, only a few key papers are reviewed here. Kabe (1985) developed a method for stiffness matrix adjustment using modal test data and preserved structural connectivity information of a mass-spring system. Stubbs (1990), in two companion papers, successfully used the frequency response of NDT data for damage evaluation of a cantilevered beam. Fritzen and Zhu (1991) used measured transfer functions to update design parameters by exciting mechanical models by broad-band impulse spectrums using an impact hammer. The FEM models of real structures were successfully updated using frequency domain data. Smith and Beattie (1991) developed a method for the optimal estimation of model parameters using inconsistent modal test data for large space structures. Gornshetyn (1992) used selected frequencies and incomplete mode shape measurements for optimal parameter estimation at the element level. Akkan et al. (1993) also used modal test data for the identification of a three-dimensional (3D) FEM of a three-span continuous bridge. They successfully correlated the FEM and the test data for bridge load rating. Strains do not relate directly to mode shapes, but Yao et al. (1992) developed a damage detection method using vibration signature analysis and the concept of "strain mode shapes." Locations and magnitudes of minor structural damage were detected for a steel structure. Although their dynamic work does not apply directly to this paper, Yao et al. did state that from their experiments, strain measurements were more sensitive to local damage and were able to identify damage much better than displacement measurements. The concept of strain mode shapes was first introduced by Li et al. (1989).

There is limited work done using static test data. Hajela and Soeiro (1990) developed parameter estimation methods using both static and modal data. Simulated measurements of static deflections or vibration modes were successfully used for pa-
parameter estimation. Sanaye and Nelson (1986) proposed a method of parameter identification using static test data measured at a subset of degrees of freedom (DOFs). Structural stiffnesses were successfully estimated at the element level, including large changes and even element failures for stable structures. This method was limited to force and displacement measurements at the same DOFs. This limitation was removed by Sanaye and Onipede (1991) using static applied forces at one subset of DOFs and measured displacements at another subset of DOFs for parameter estimation. These two subsets may or may not overlap. Hjelmslad has also done extensive work on the parameter estimation of structures using static test data. He is the coauthor of companion papers by Banan et al. (1994). They use incomplete sets of applied static forces and displacements to estimate element stiffnesses. In this method, the unmeasured displacements are also assumed to be unknown and estimated. On one hand, this assumption simplifies the error function to be minimized; on the other hand, the number of unknowns is increased to the unknown parameters plus the unmeasured displacements. Bruno (1994) formulated a parameter identification technique to locate and characterize loose joints of a deployable space truss using actuator-induced static loading and unloading.

The major goal of this paper is to develop a method for the parameter estimation of linear-elastic structures using static strain measurements, preserving structural connectivity. This method allows one single or several static forces to be applied at a subset of DOFs, and strains to be measured at a subset of structural components. In addition, it is possible to identify all or a portion of structural cross-sectional properties, including element failures. The feasibility of the proposed method is numerically demonstrated for truss and frame types of structures, and the cross-sectional parameters are successfully identified by the method.

The parameter estimation method presented here is an excellent tool for studying the behavior of structures, and is expanded [in a companion paper (Sanaye and Saletnik 1996)] to a tool for the design of nondestructive tests for parameter estimation and damage assessment using noisy measurements. It is paramount to preselect a subset of applied forces and a subset of measured strains leading to a noise-tolerant parameter estimation system prior to any nondestructive testing and data acquisition. This is a combinatorial problem creating an enormous number of possible combinations of measurements, which requires a systematic technique for the pretest selection of a subset of measurements leading to a successful identification.

**PARAMETER ESTIMATION FORMULATION**

An optimization-based parameter estimation method is presented to adjust the parameters of a finite element model with simulated static strain measurements. Forces are applied at a subset of DOFs and strains are measured on a limited number of structural elements. By using the Gauss-Newton method or the steepest descent method, an iterative approach is established to solve for the structural stiffnesses at the element level. A detailed discussion is available in Saletnik (1993).

**Strain-Displacement Relations**

The finite element model is based on the stiffness relationship between forces and displacements. To utilize strain measurements, a mapping between displacements and strains must first be developed. The model for this mapping is derived from the geometric relationship of nodal displacements to elemental strains. This requires different relations for axial elements (truss members), than for axial and bending elements (frame members). This relationship will be created in the form of an element mapping vector \( \mathbf{B}_e \) in the global coordinates.

The first step is to formulate the element mapping vector \( \mathbf{B}_e \) in the local coordinates such that the following relation is satisfied for an elemental strain \( \mathbf{e}_e \) and displacements \( \mathbf{U}_e \):

\[
\mathbf{e}_e = \mathbf{B}_e \{ \mathbf{U}_e \}
\]

then \( \{ \mathbf{U}_e \} \) is transformed from the local coordinates to the global coordinates as

\[
\{ \mathbf{U}_e \} = [\mathbf{T}_e] \{ \mathbf{U}_e \}
\]

where \([\mathbf{T}_e]\) = mapping matrix; and \( \{ \mathbf{U}_e \} \) = element nodal displacements in the global coordinates. By substituting (2) in (1), the global mapping relation for one structural element is determined as

\[
\mathbf{e}_e = \{ \mathbf{B}_e \} \{ \mathbf{U}_e \}
\]

where \( \{ \mathbf{B}_e \} \) is

\[
\{ \mathbf{B}_e \} = [\mathbf{B}_e][\mathbf{T}_e]
\]

Given a system of \( \text{NEL} \) elements \( n \), the system \( \mathbf{B} \) matrix is assembled by vertically augmenting the element strains and aligning system DOFs horizontally as

\[
\{ \mathbf{e} \} = \{ \mathbf{B} \} \{ \mathbf{U} \}
\]

Now, \( \{ \mathbf{e} \} \) = elemental strain vector of size \( \text{NEL} \times 1 \); and \( \{ \mathbf{U} \} \) = global displacement vector. This assembly will produce the matrix \( \{ \mathbf{B} \} \) of size \( \text{NEL} \times NDOF \). However, it will be shown later that only a portion of this matrix is necessary.

**Creating \( \{ \mathbf{B}_e \} \) for Truss Element**

Fig. 1 illustrates a generic truss element. The definition of axial strain in terms of the axial DOFs is

\[
\varepsilon_a = \frac{u_i - u_j}{L_e}
\]

where \( u_i \) and \( u_j \) = displacements at nodes \( i \) and \( j \) in the local element coordinate system; and \( L_e \) = length of element \( ij \). Then, (6) may be arranged into matrix form as

\[
\varepsilon_a = \frac{1}{L_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}
\]

On comparing (7) and (1), the inner term is extracted as the mapping vector \( \mathbf{B}_e \) for the axial deformation of a truss element \( n \) in the local coordinates.

![Fig. 1. Truss Element](image-url)
\[ \{ \bar{B}_n \} = \frac{1}{L_n} \begin{pmatrix} -1 & 1 \end{pmatrix} \]  

To transform the mapping matrix from the local coordinates \( \mathbf{x} \) to the global coordinates in a 2D or 3D space, direction cosines are defined as in Fig. 2. Mathematically, they are

\[ \begin{align*}
    l_{x} &= \cos(\alpha_x); \\
    l_{y} &= \cos(\alpha_y); \\
    l_{z} &= \cos(\alpha_z)
\end{align*} \]

where \( l_{\mathbf{x}} \) = cosine of the angle between the local \( \mathbf{x} \)-axis and the global \( \mathbf{X} \)-axis.

For transforming a truss element from the local coordinates to a 2D space, \( l_{\mathbf{x}} \) is set equal to zero. The relation between the local displacements \( \{ \bar{U}_n \} \), global displacements \( \{ U_n \} \), and transformation matrix \( \{ T_n \} \) in (2) is

\[ \begin{pmatrix} \bar{u}_i \\ \bar{v}_i \end{pmatrix} = \begin{pmatrix} l_{x} & l_{y} & 0 & 0 \\ 0 & 0 & l_{x} & l_{y} \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix} \]

This will give \( \{ B_n \} = \{ \bar{B} \} \{ T_n \} \) of expression (4), where sizes are \( 1 \times 4, 1 \times 2, \) and \( 2 \times 4 \), respectively.

By combining the definitions of (7) and (10), the 2D truss strain-displacement relation is derived

\[ \varepsilon_n = \frac{1}{L_n} \begin{pmatrix} -l_{x} & -l_{y} & l_{x} & l_{y} \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix} \]

On comparing (11) and (3), the inner term is extracted as the mapping vector \( \{ B_n \} \) for the axial deformation of a truss element \( n \) in the 2D global coordinates

\[ \{ B_n \} = \frac{1}{L_n} \begin{pmatrix} -l_{x} & -l_{y} & l_{x} & l_{y} \end{pmatrix} \]

This is extended to three dimensions by including the third direction cosine as

\[ \{ B_n \} = \frac{1}{L_n} \begin{pmatrix} -l_{x} & -l_{y} & -l_{z} & l_{x} & l_{y} & l_{z} \end{pmatrix} \]

Then, using (5), \( \{ \xi \} \) is assembled for a structural system of NEL elements \( n \).

**Creating \( \{ B_n \} \) for Frame Element**

When creating the mapping vector \( \{ B_n \} \) for the frame element, the actions of axial deformation and bending both cause axial strains on the element surface at a known distance from the neutral axis. For linear-elastic behavior, these two actions are superimposed. These strains are measured in the \( \mathbf{x} \) (lengthwise) direction. Strain gauges can be applied in this direction for data acquisition leading to parameter estimation. A generic 2D frame element is shown in Fig. 3. There are six DOFs: two translations and one rotation at each node.

First, the effect of bending is established. The relevant DOFs are \( \bar{\psi} \) and \( \bar{\nu} \) at both nodes \( i \) and \( j \). These provide a translation and a rotation at each node, and are used in the definition of a surface strain in the local coordinates

\[ \varepsilon_n = -\frac{y}{L} \frac{d^2 \bar{\psi}}{dx^2} \]

where \( x \) = axial distance from node \( i \); \( \bar{\psi} \) = translation in the \( \bar{y} \) direction; and \( \bar{y} \) = distance from the neutral axis to the strain measurement surface. For finite element analysis a natural coordinate system \( \xi \), which ranges from \( -1 \) to \( +1 \) at node \( i \) to \( +1 \) at node \( j \), is defined to substitute \( x \) that ranges from \( 0 \) to \( L \)

\[ \xi = \frac{2x}{L} - 1 \]

Then, Hermite shape functions are used to model the pure bending behavior

\[ \begin{align*}
    H_1 &= \frac{1}{4} (1 - \xi)^2 (2 + \xi); \\
    H_2 &= \frac{1}{4} (1 - \xi)^2 (\xi + 1)
\end{align*} \]

\[ \begin{align*}
    H_3 &= \frac{1}{4} (1 + \xi)^2 (2 - \xi); \\
    H_4 &= \frac{1}{4} (1 + \xi)^2 (\xi - 1)
\end{align*} \]

Each function represents bending deformation due to a unit displacement at a given DOF. The bending deformation is represented in the local coordinate system as

\[ \bar{\psi}(\xi) = H_i \bar{\psi}_i + \frac{L}{2} H_{ii} \bar{\psi}_i + H_{ij} \bar{\psi}_j + \frac{L}{2} H_{ij} \bar{\psi}_j \]

The strain expression (14) is in terms of \( \xi \), but the Hermite shape functions are in terms of \( \xi \). Therefore, (14) is expanded in terms of \( \xi \) as

\[ \varepsilon_n = -\frac{1}{L} \frac{d^2 \bar{\psi}}{dx^2} \]

JOURNAL OF STRUCTURAL ENGINEERING / MAY 1996 / 557
Eq. (17) is substituted into the strain expression (18) to give

$$\varepsilon_n = -\frac{\gamma}{L^2} \frac{d^2 \{H\}}{d\xi^2} \begin{Bmatrix} \hat{\gamma} \\ \hat{\xi} \end{Bmatrix}$$

(19)

where \(\{H\}\) is

$$\{H\} = \begin{Bmatrix} H_1 \\ \frac{L}{2} H_2 \\ H_3 \\ \frac{L}{2} H_4 \end{Bmatrix}$$

(20)

At this point, (19) is in terms of the second derivatives of \(\{H\}\) with respect to \(\xi\). Since (19) has strain on the left and displacements on the right, the middle is compared with (1) and extracted as the mapping function \(\{B_n\}\) for the pure bending of an element \(n\)

$$\{B_n\} = \frac{-\gamma}{L^2} \begin{Bmatrix} 6\xi & \frac{L}{2} (6\xi - 2) \\ -\frac{L}{2} (6\xi + 2) \end{Bmatrix}$$

(21)

Now, axial effects need to be accounted for. The relevant DOFs are \(\bar{u}, \bar{v}\) and \(\bar{w}\). All 6 DOFs have been accounted for by (8) and (21). Putting these together by superposition forms

$$\{B_n\} = \begin{Bmatrix} -1 \\ -\frac{6\xi}{L^2} \frac{L}{2} (6\xi - 2) \frac{6\xi}{L^2} \frac{L}{2} (6\xi + 2) \end{Bmatrix}$$

(22)

On recalling (2), a transformation matrix is still needed to convert the local mapping matrix to the global 2D coordinate system. The relation between the local displacements \(\{U_n\}\), global displacements \(\{U\}\), and transformation matrix \(\{T_n\}\) is

$$\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix} = \begin{Bmatrix} l_x & l_y & l_z \\ l_x & l_y & l_z \\ l_x & l_y & l_z \end{Bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

(23)

This will give \(\{B_n\} = \{B\} \{T_n\}\), where sizes are \(1 \times 6, 1 \times 6, \) and \(6 \times 6\), respectively. \(\{B_n\}\) is then assembled into \(\{B\}\) as in (5). It should be noted that frame and truss elements can be mixed in the structure being modeled. The assembly of \(\{B\}\) is transparent to the native types of the \(\{B_n\}\) submatrices, and by superposition will exhibit valid behavior.

The extension of this method to the 3D frame element is a straightforward task. Since strains are assumed to be measured in the \(X\) direction only, strong-axis bending and weak-axis bending are independent and can be evaluated separately and superimposed. There are six DOFs at each node, corresponding to the three axes of translation followed by rotations about those axes. The location of the strain surface is now defined by both \(\bar{\gamma}\) and \(\bar{\xi}\). The effect of twist on surface axial strains is neglected. Although complex cross sections can have lengthwise surface strains due to warping caused by applied torsion, a nonwarping section is assumed. Thus the weak-axis terms are treated similar to the strong-axis terms, but with \(\bar{\xi}\) instead of \(\bar{\gamma}\)

$$\{B_n\} = \begin{Bmatrix} -1 \\ -\frac{6\xi}{L^2} \\ 0 \\ -\frac{L}{2} (6\xi - 2) \\ 0 \\ -\frac{L}{2} (6\xi + 2) \end{Bmatrix}$$

(24)

The transformation matrix for a frame element in 3D space becomes

$$\{T_n\} = \begin{Bmatrix} l_x & l_y & l_z & 0 & 0 & 0 \\ l_x & l_y & l_z & 0 & 0 & 0 \\ l_x & l_y & l_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & l_y & l_z \\ 0 & 0 & 0 & l_x & l_y & l_z \\ 0 & 0 & 0 & l_x & l_y & l_z \end{Bmatrix}$$

(25)

This will give \(\{B_n\} = \{B\} \{T_n\}\), where sizes are \(1 \times 12, 1 \times 12, \) and \(12 \times 12\), respectively. This transformation matrix differs from that of the truss element due to the fact that all three local axes need to be translated. In the case of the truss element, only the local \(X\) axis was defined. For the frame element, \(X, Y,\) and \(Z\) are all taken relative to the \(X, Y,\) and \(Z\) axes. Then, using (5), \(\{B\}\) is assembled for a structural system of \(NEL\) elements \(n\).

Finite-Element Model

The static finite element equation for a constrained structural system is

$$\{F\} = [K(p)]\{U\}$$

(26)

by substituting (26) in (5) it gives

$$\{\varepsilon\} = [B][K(p)]^{-1}\{F\}$$

(27)

where \(\{\varepsilon\} = NEL \times 1\); \([B] = NEL \times NDOF, [K(p)] = NDOF \times NDOF;\) and \(\{F\} = NDOF \times 1\). The element stiffness parameters enter the picture through the stiffness matrix \([K(p)]\). These parameters may be a cross-sectional area \(A\), or a moment of inertia \(I_{xx}, I_{yy},\) or \(I_{zz}\) and stored in the vector of parameters \(\{p\}\). For the parameter identification method, however, \(NSF\) load cases are to be used in solving for the unknown parameters. These test data are horizontally concatenated onto \(\{\varepsilon\}\) and \(\{F\}\), giving

$$\{\varepsilon\} = [B][K(p)]^{-1}\{F\}$$

(28)

where \(\{\varepsilon\} = NEL \times NSF;\) and \(\{F\} = NDOF \times NSF\).

It is not required to measure all the strains of a system; therefore, (28) is first partitioned based on measured strains \(a\) and unmeasured strains \(b\)

$$\begin{Bmatrix} \varepsilon_a \\ \varepsilon_b \end{Bmatrix} = [B_a][K(p)]^{-1}\{F\}$$

(29)

Since there is no need for unmeasured strains, \(\varepsilon_b\) is eliminated as

$$\varepsilon_a = [B_a][K(p)]^{-1}\{F\}$$

(30)

where \(\varepsilon_a = NMS \times NSF;\) \(B_a = NMS \times NDOF;\) and \(NMS = number\ of\ measured\ strains.\ Structural\ elements\ included\ in\ the\ measured\ category\ may\ have\ one\ or\ more\ strain\ gauges\ per\ element\ mounted\ at\ strategic\ locations.\ Eq.\ (30)\ describes\ the\ relationship\ between\ strains,\ forces,\ and\ parameters,\ and\ will\ be\ used\ to\ solve\ for\ those\ parameters\ \{p\}\ of\ size\ NUP \times 1\).

Output Strain Error Function

To measure the difference between the analytical strains and measured strains, an error function is formed. The “output strain error function” is defined as
\[ [e(p)] = [e_s(p)]^n - [e_s]^n \]  
(31)

The superscript = analytical values; and \( m \) = measured values. On substituting (30) in (31)

\[ [e(p)] = [B_s][K(p)]^{-1}[F] - [e_s]^n \]  
(32)

The error function is of size \( NMS \times NSF \), and, due to the inversion of \([K(p)]\), is a nonlinear function of the parameters \( p \). To linearize (32) using Taylor series expansion, the error function matrix is vectorized by concatenating all the columns vertically. This produces an \([e(p)]\) of size \( NM \times 1 \), where the number of measurements \( NM = NMS \times NSF \). Then \([e(p)]\) is linearized using a first-order Taylor series expansion as

\[ [e(p + \Delta p)] = [e(p)] + \left[ \frac{\partial [e(p)]}{\partial [p]} \right] \Delta p \]  
(33)

This sets up the definition of the sensitivity matrix \([S(p)]\) of size \( NM \times NUP \) as

\[ [S(p)] = \left[ \frac{\partial [e(p)]}{\partial [p]} \right] \]  
(34)

To form the sensitivity matrix \([S(p)]\), first differentiate \([e(p)]\) in (32) with respect to one parameter \( p_i \) to form \([S(p)]\) of size \( NMS \times NSF \)

\[ [S(p)] = -[B_s][K(p)]^{-1} \cdot \frac{\partial}{\partial p_i} ([K(p)][K(p)]^{-1}[F]) \]  
(35)

Then similar to the vectorization of \([e(p)]\), \([S(p)]\) is unrolled vertically into a vector, \([S(p)]\), which is one column of the matrix \([S(p)]\) in (34). These vectors are concatenated column-wise to form the full sensitivity matrix \([S(p)]\).

**Minimization of Error Function**

The error function \([e(p)]\) is reduced to a scalar performance error function, \( J(p) \) by taking the Euclidean norm of the error vector

\[ J(p) = [e(p + \Delta p)]^T[e(p + \Delta p)] \]  
(36)

To minimize the performance error function with respect to the unknown parameters \( p \), subjected to \( p_i = 0 \) for \( i = 1 \) to \( NUP \), the gradient of \( J(p) \) with respect to \( \Delta p \) is set to zero. This results in a linearized system of equations as

\[ [S]^T[S][\Delta p] = -[S]^T[e(p)] \]  
(37)

Where \( \Delta p \) = change in parameters and is of size \( NUP \times 1 \). This system of equations may be solved by several methods.

**Solution Techniques for \( \Delta p \)**

There are several ways to solve (37) for \( \Delta p \). The goal of the parameter identification technique, however, is to use as few measurements as possible. If the number of measurements is equal to the number of unknown parameters \( NM = NUP \), \([S]\) is square and direct inversion may be used

\[ \{\Delta p\} = -[S]^{-1}[e(p)] \]  
(38)

If \( NM > NUP \), then \([S]\) will not be square. For these cases, the least-squares method results in

\[ \{\Delta p\} = -(S^T[S])^{-1}[S]^T[e(p)] \]  
(39)

For both (38) and (39), \([e(p)]\) is defined by the unrolled form of (31), and \([S]\) is defined by (34). Then, \([\Delta p]\) will be used to set up an iterative solution for \( p \) of size \( NUP \times 1 \) to minimize \( J(p) \). The preceding nonlinear regression method is based on determining the values of parameters that minimize the sum of the squares of the residuals and is categorized as the Gauss-Newton method of solution of a nonlinear system of equations.

Depending on the topology of the performance error function, \([S]^T[S]\) may be singular or ill-conditioned. In these cases, there will be eigenvalues of \([S]^T[S]\) that are near or equal to zero. Examination of a row-echelon decomposition of (37) may show nonzero elements in the upper triangle of \([S]^T[S]\). This rank deficiency indicates a linear dependency between two or more parameters for the given measurements. These parameters cannot be determined from the provided measurement data. However, it is possible to use a technique known as singular-value decomposition to solve for the linearly independent parameters of the system of equations. The dependent parameters are assigned \( \Delta p = 0 \).

It is possible to approach the iterative solution differently from how it is presented here. The gradient or steepest-descent method of optimization is also used for the parameter identification solution. One advantage of the gradient method is that it is not susceptible to divergence, as opposed to the Gauss-Newton method. A small shape discrepancy can cause the Gauss-Newton method to produce a discrepant \( \Delta p \), and cause divergence even though the system has a solution. The gradient method, however, has drawbacks. It will always find a minimum, but that is not necessarily the global minimum. Additionally, the gradient method has a linear rate of convergence, whereas the Gauss-Newton method has a quadratic asymptotic rate of convergence, which is much faster. For these reasons, a technique is developed to begin iterating by the gradient method and then switching to the Gauss-Newton method once the solution has progressed close enough to the solution. For systems that are sensitive to divergence by the Gauss-Newton method, this is a practical approach to a solution.

**Solving for Parameters**

Once \( \Delta p \) has been established by solving (37), an iterative technique is used, whereby, for each iteration \( k \)

\[ \{p_{i+1}\} = \{p_i\} + \{\Delta p\} \]  
(40)

There are several possible convergence conditions for parameter identification. The performance error function \( J_n \), the maximum value of the error vector \( [e_n] \), the maximum change of any of the parameters \( \Delta p_n \), or the maximum relative change in parameters \( (\Delta p_n)/p_i \) are all candidates for comparison with predefined limits. If none of these values meets its prescribed limit before a certain number of iterations, it is assumed that the system has diverged.

As stated before, the number of independent measurements \( NM = NMS \times NSF \). If \( NM > NUP \) there may exist a unique solution to (37). However, this is just the necessary condition for the uniqueness of the solution, not the sufficient condition. In the two examples presented in this paper, using simulated static test data, nonunique solutions were never encountered.

**PARAMETER ESTIMATION EXAMPLES**

Before any algorithm can be expected to estimate the parameters of a finite element model from measured data, it must perform adequately when given noise-free data. Using simulated data with no measurement noise, the algorithm must identify the true values of parameters. An initial value of parameters \( \{p_i\} \) is used to begin the iterative process. The measurement data is simulated using the true value of parameters \( \{p_i\} \). The algorithm should satisfactorily iterate \( \{p_i\} \) to \( \{p_i\} \).

To carry out this evaluation, two example structures are used: a two-dimensional truss and a two-dimensional frame. For each example, several arbitrary combinations of force and
Truss Example

The first example structure is a two-dimensional truss, as seen in Fig. 4. It is modeled as a true truss, with members carrying axial loads only and pinned at the ends. This also means that the only applicable parameter is the cross-sectional area $A$ of the members. Strain measurements are simple: there is one strain per element. This truss has 10 elements; therefore, there are 10 strain measurements and 10 parameters ($NUP = 10$). The physical properties are defined as follows:

- Modulus of elasticity $E$ for all elements = 30,000 ksi (206.8 GPa)
- Initial iteration value of parameters $p_i = 5.0$ sq in. (32.26 cm$^2$)
- True value of parameters $p_i = 3.0$ sq in. (19.35 cm$^2$)

Ten experiments are carried out and presented in Table 1. For each experiment, a 100 lb (445 N) load is applied to each selected $FDOF$, one at a time. Therefore, each load set contains only one force (this method is capable of using multiple forces per load case). Strains, $e$, are measured on each selected element for each load set. The number of measurements, $NM$, must be greater than or equal to the number of unknown parameters, $NUP$, as a necessary condition for a solution to exist. If the rank of the system of equations is equal to $NUP$, then a solution exists. However, this is not a sufficient condition. The Gauss-Newton method of solution was used to exploit the quick rate of convergence as reported in Table 1, column 8. All 10 cross-sectional areas are assumed to be unknown, except for cases 3 and 8, which use the same $FDOF$ and $e$ as cases 2 and 7, respectively. Since no noise was included in the simulated measurements, all parameters converged to the true values of 3,000 with the accuracy of $10^{-4}$.

Eq. (37) is solved for $\delta p$ during each iteration. The rank is calculated from the number of independent rows when using elementary row operations on (37) to reduce it to row-echelon form. “Singular” indicates that the algorithm was unable to solve due to a singularity in $[S]$. A rank deficiency (rank < $NUP$) is an additional indicator of linear dependencies. Those dependent parameters do not have a unique solution based on the provided information. The truss behaves best when all elements are stressed through an experiment that consists of several loadings. Therefore, a case with appropriate forces and a few strain measurements is ideal.

Any of the cases that failed, i.e., 2 and 7, can be solved either by assuming the dependent parameters to be known, or by using singular value decomposition (Saletnik 1993). Although these methods will result in only partial solutions, they show that nearly any load case can be used. Clearly, however,
some cases are better than others. If a nondestructive test is going to be performed, instead of using singular or ill-conditioned cases it is better to select cases that are well-behaved. It is also shown [in Sanayei and Saletnik (1996)] that ill-conditioned cases perform poorly in the presence of measurement noise. However, the intent is to move away from such cases and find well-conditioned cases that can tolerate some degree of measurement noise.

Frame Example

The second example structure is a 2D frame, as seen in Fig. 5. This structure is composed of frame elements, capable of bending as well as axial deformation, and thus having two parameters: the cross-sectional area $A$ and the moment of inertia $I$. Since there are seven elements, there can be up to 14 unknown parameters ($NUP = 14$). The modulus of elasticity $E$ for all elements is 30,000 ksi (206.8 GPa) and the cross-sectional properties are presented in Table 2.

There are 10 cases for the parameter identification of the 2D frame example. In all cases, the applied forces are 100 lb (445 N) or 100 ft-lb (136 N·m), as appropriate. The results are summarized in Table 3. This table indicates that the 2D frame is solvable for every case except case 6. Both parameters $A$ and $l$, or a subset thereof, can be solved for, using a variety of different measurements and forces. All cases that converged identified the true values of parameters, $p$, in Table 2, with the required accuracy of $10^{-4}$.

CONCLUSIONS

A new method was proposed and successfully verified for parameter identification at the element level using static applied forces at a subset of DOFs and strain measurements at selective locations for linear-elastic structures. A quadratic performance error function was formed using the difference between the analytical and measured strains. These strains represent both axial and bending deformations. The performance error function was minimized to identify the stiffness parameters at the element level. Equivalent cross-sectional element properties such as areas for trusses and areas and/or moments of inerts for frames were identified. This method is capable of identification of all or a selective subset of parameters of structures, including parameters with zero values in case of failures.

Strain measurements using strain gauges are more accurate and much simpler than displacement measurements. They also do not require a frame of reference, which makes them more attractive for large-scale testing. For the proposed method, element strains are mapped from measured displacements and substituted into the finite-element-based error function. When using axial-only elements, there is a direct mapping based solely on the orientation of the element relative to the coordinate system. When bending is included, surface strains are calculated based on their distance from the neutral axis. This calculation is valid both for strong and weak axis bending. Through superposition, both bending effects and the axial effect are combined into one strain measurement along the axial direction of the element.

Two numerical examples were examined to validate the proposed method. It was demonstrated that with the use of simulated noise-free subsets of force and strain measurements, the "exact" values of equivalent cross-sectional parameters are successfully identified. When the number of measurements is greater than or equal to the unknown parameters, it is possible to perform a parameter identification. However, this is just the necessary condition and not the sufficient condition for successful parameter identification. If linear dependencies exist in the sensitivity matrix, they reduce the capability to identify all parameters simultaneously. In these cases either another subset of measurements must be used or only a partial set of parameters identified. In the case of ill-conditioned sensitivity matrices, it is advisable to change the subset of measurements to avoid large errors in the identified parameters. In terms of uniqueness of the identified parameters, there is no mathematical proof ensuring that the identified parameters are unique. However, based on the experience of the writers with the proposed method using static simulated data, when the algorithm converged, it converged on the correct values of the parameters and not on a local minimum. Also, most of the computer runs, using reasonably selected subsets of measurements, converged. However, a few cases did not run due to a singular or ill-conditioned sensitivity matrix.

Future work can include laboratory testing to determine the behavior patterns of various types of failures and to validate the proposed method experimentally, allowing this technique to be expanded outside of the laboratory to a full damage assessment system. Additional research can include expansion of the algorithm to work with dynamic excitation and strain measurements to create a reference-independent estimation technique.

APPENDIX. REFERENCES


