PARAMETER ESTIMATION OF STRUCTURES FROM STATIC STRAIN MEASUREMENTS. II: ERROR SENSITIVITY ANALYSIS

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ABSTRACT: A method for the parameter estimation of structures was developed in a companion paper. It used subsets of static applied forces and strain measurements and successfully identified structural element stiffnesses. In the presence of noisy measurements, the selection of these subsets drastically affects the accuracy of the parameter estimates. In addition, there are a large number of combinations of applied force and strain measurements to be considered. Therefore, a heuristic method is presented for choosing a small subset of forces and strain measurements so as to have the least sensitivity to measurement noise. Once chosen, using a Monte Carlo analysis the subset of measurements is analyzed to determine an input-output error relationship. This heuristic method can be used to design near-optimal experiments for the nondestructive testing and parameter estimation of structures. A truss and a frame example are presented and a small and noise-tolerant subset of forces and strains is successfully selected for possible nondestructive testing.

INTRODUCTION

A companion paper by Sanayei and Saletnik (1996) presented a method for the parameter estimation of linear-elastic structures using applied static forces and strain measurements. Using a subset of applied forces and a subset of measured strains, element stiffnesses of all or a portion of the structure were identified. Simulated measurements were noise-free resulting in "exact" identified stiffnesses. If actual nondestructive test (NDT) data is going to be used, the effect of measurement noise on the parameter estimates must be studied.

As with all measurement devices, strain gauges are subject to measurement noise. In the presence of noisy measurements (known as input error) the parameter estimation algorithm will give results different from those with noise-free simulated data. The difference between the estimated parameters and the true values (known as output error) is evaluated to determine the impact of input errors for a specific subset of measurements. Monte Carlo simulations are performed to study and compare the input-output error behavior of various subsets of measurements. It is desirable, for labor and economic reasons, to use as few inputs (applied forces and strain measurements) as necessary. However, each set of input has its own input-output error relationship; some sets of measurements are more sensitive to errors than others. Based on the writers' experience, a random or judgmental selection is not typically sufficient.

There are several theories on how to achieve the optimum subset of measurements for parameter estimation; they are mostly rooted in the active vibration control theories. Haftka and Adelman (1985) proposed two integer programming methods for the selection of actuator locations to correct surface distortions of an orbiting spacecraft. These methods are compared with the heuristic method proposed by DeLorenzo, an improved version of which was published in 1990. DeLorenzo's sensor and actuator configuration was used to control large space structures. Kammer (1991) developed the method of "effective independence" for the selection of a set of sensor locations for on-orbit modal identification. Modal kinetic energy distribution was used to rank the contributions of each candidate sensor location to target mode shape. Holnicki-Szulc et al. (1992) also developed a method called the "progressive collapse analog" for optimal locations of actuators controlling the selected modes of vibration.

Lim (1991) determined that damage is easiest to detect for those elements that are fully participating in load bearing and contain the most energy. When such an element is damaged, it is well-represented in the estimation results by a notable difference between the expected and estimated parameters. Elements that are not participating in load bearing and thus do not have large amounts of contained energy do not show this variation. Rather, there is an overall change in the estimated parameters of the system; individual element damage cannot be picked out. Hajela and Soeiro (1990b) presented the idea of "dominant displacements" for both static and vibrational testing, whereby certain forces and measurements are more representative of the structural system. They show also that errors are more prevalent when loading does not result in an equal stress distribution in each of the members. In a related paper, Hajela and Soeiro (1990a) show through experimental testing that uniform stress loading produces excellent results.

There is little research using static test data for parameter estimation. Jhembstad developed a method of parameter estimation using static force and displacement measurements. He expanded his research by studying the input-output error behavior of a linear-elastic static system in a paper by Banan et al. (1994). Noticing the large variation of the input-output error behavior based on the selection of the subset of measurements, it is necessary to establish a method to select the most promising subsets of measurements to ensure a successful parameter estimation. There are a large number of possible subsets of measurements to be considered for NDT (Sanayei 1992). Sanayei et al. (1992) studied the impact of measurement noise on parameter estimates and proposed a heuristic method for the pretest selection of static forces and displacement measurements.

The main goal of this paper is to study the input-output error behavior of the parameter estimation method presented in Sanayei and Saletnik (1996), and to design a successful NDT. It is known that the effectiveness of a parameter estimation system is strongly dependent on the measurement locations and number of measurements. A heuristic method based on error sensitivity analysis is proposed for reducing the number of force and strain measurements, from a starting point of using all forces and measurements to a final small subset that is noise-tolerant. This heuristic procedure is called the best-in-worst-out (BIWO) method and is used for the pretest design of nondestructive experiments in the presence of measurement noise, hereon referred to as experiment design. This method...
selects a small subset of noise-tolerant measurements to ensure a successful parameter estimation using actual NDT data. Although the BIWO method used for the experiment design is computation-intensive, it is straightforward and can be directly programmed to require no human interaction. After the selection of the target measurements, only one NDT and one parameter estimation run is required. A truss and a frame example are used to illustrate the preceding method. One of the numerical examples shows a lower output error for a subset compared to using the initial full set of forces and strains. This shows that a well-designed nondestructive experiment can be used effectively and reliably for parameter estimation; a prerequisite for the damage assessment and health monitoring of structures. There is no guarantee that the result of the BIWO method will be optimum; however, it leads to a near-optimal and noise-tolerant system acceptable for all practical purposes.

PARAMETER ESTIMATION IN PRESENCE OF MEASUREMENT NOISE

The companion paper showed that the error function introduced, shown here in the following, does not contaminate the identified parameters if noise-free simulated measurements are used:

\[ [e(p)] = [e_r(p)] - [e_m] = ([B_e][K(p)][U] - [F] - [e_m]) \]  

(1)

The proposed strain output error function, \([e(p)]\), is based on minimizing the difference between a subset of analytical strains, \([e_r(p)]\), and the measured strains, \([e_m]\), to estimate NUP unknown stiffness parameters. The analytical strains are based on finite element analysis using the basic relation \([F] = [K(p)][U]\) relating static forces and displacements through the stiffness matrix. To generate these measurements, NSF sets of forces are applied at NDOP degrees of freedom and NMS sets of measured strains are made for a subset of structural elements. Also, \([B_e]\) is the strain-displacement mapping matrix. For detailed description and derivation of this error function refer to Sanaye and Saletnik (1996).

The first test for a parameter identification algorithm is to see how well it performs when given exact data. As was shown in the companion paper, no measurement noise leads to no identification error. In the real world, however, measurements are never exact. The goal is to devise an approach for the design of NDT, but once that is achieved, it will be advantageous to know how identified parameters react to noise in certain measurements. Thus, a method for modeling measurement noise must first be developed.

Input Error

Any mechanical or electronic device has a range of measurements beyond which measurements are uncertain. Even within that range errors can occur for a variety of reasons, including human error. Noise in the applied forces or the measured strains is known as input error. Strain gauges are delicate devices; if not properly affixed they will give erroneous results. Additionally, misalignment with the local centroidal axis, \(X\), can lead to errors. The transverse distances to the centroidal axis in the local coordinate system (i.e., \(y\) and \(z\)) are also critical. Additionally, once the gauge is functional, there are potential errors in reading it. Wire and solder resistance, the accuracy of the strain measuring device, and proper balancing and zeroing of the circuitry are all crucial to obtaining proper readings. In addition to the possible errors in strain measurement, there can be errors in the applied force measurements.

There are other ways in which errors may propagate in the parameter estimation algorithm. The finite element model may not be accurate in describing the physical model (e.g., element behavior, connections, and support conditions). The modeling error, as it is known, may also include other effects such as manufacturing inconsistencies, residual or thermal stresses, or material flaws. Modeling error is not the topic of this discussion, and is not considered in this paper.

Modeling of Input Error

With all these possibilities for introducing noise into a system, it is difficult, if not impossible, to mathematically model measurement noise. However, for numerical experimentation, it can be done by varying the force and strain measurement values slightly with a known probability distribution. There are a number of ways to model the input error, \(I_e\). The most commonly used distributions are the uniform and normal distributions.

The uniform distribution can represent an equal probability at any one time ranging from \(-I_e\) to \(+I_e\). The variance of measurement noise is

\[ \sigma^2 = \frac{I_e^2}{3} \]  

(2)

Eq. (2) is used to generate a set of uniform random numbers with a mean of zero and a variance of \(I_e^2/3\).

The normal distribution, often known as the bell curve or Gaussian, represents a higher probability of noise level closer

<table>
<thead>
<tr>
<th>Element</th>
<th>(p_i)</th>
<th>(p)</th>
<th>Percent error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0</td>
<td>2.973</td>
<td>0.90</td>
</tr>
<tr>
<td>2</td>
<td>3.0</td>
<td>2.983</td>
<td>0.57</td>
</tr>
<tr>
<td>3</td>
<td>3.0</td>
<td>3.414</td>
<td>13.80</td>
</tr>
<tr>
<td>4</td>
<td>3.0</td>
<td>2.948</td>
<td>1.73</td>
</tr>
<tr>
<td>5</td>
<td>3.0</td>
<td>2.722</td>
<td>9.27</td>
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<td>3.0</td>
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<td>1.23</td>
</tr>
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<td>7</td>
<td>3.0</td>
<td>2.994</td>
<td>0.20</td>
</tr>
<tr>
<td>8</td>
<td>3.0</td>
<td>2.902</td>
<td>3.27</td>
</tr>
<tr>
<td>9</td>
<td>3.0</td>
<td>2.957</td>
<td>1.43</td>
</tr>
<tr>
<td>10</td>
<td>3.0</td>
<td>2.928</td>
<td>2.40</td>
</tr>
</tbody>
</table>

TABLE 1. 2D Truss Identification % Error, Case 5 with 1% Uniform Proportional Error (Case 5: Applied Forces 5–8; Measured Strains 1, 2, 6, 7)
The errors are applied proportionally to simulate contaminated parameter estimates. The percentage error level is highly variable in the estimated cross-sectional areas shown in Table 1. As an example, case 5 of Table 1 in the companion paper is now solved with a 1.0\% proportional uniform error applied to both forces and strains and converged in six iterations. The difference between parameter estimates and exact values is known as the percentage error in each parameter estimate and is reported in Table 1. When noise-free simulated measurements were used in the companion paper, the identified parameters were all 3.000; however, when using noisy measurements one should expect contaminated parameter estimates. The percentage error level is highly variable in the estimated cross-sectional areas shown in Table 1.

Two-Dimensional Frame Example with Measurement Noise

The second example structure is the same 2D frame from Sanayei and Saletnik (1996), as seen in Fig. 2. The modulus of elasticity $E$ is 30,000 ksi (206.8 GPa).

The basis of the parameter estimation algorithm is the finite element model $\{F\} = [K(p)]\{U\}$. If measured $\{F\}$ and $\{U\}$ differ at all from the exact values, then the identified parameters making up $[K(p)]$ will be different from the exact ones. This also holds true for strain measurements.

Next, in order to study the effect of the input error on the parameter estimates, the two-dimensional (2D) truss and 2D frame examples from the companion paper are used again. These two parallel examples are summarized in the following section, and used throughout this paper.

Two-Dimensional Truss Example with Measurement Noise

To demonstrate the effect of input error on parameter estimation, the 2D truss example from Sanayei and Saletnik (1996) is again used and illustrated in Fig. 1. The modulus of elasticity and cross-sectional areas are as follows:

- Modulus of elasticity $E$ for all elements = 30,000 ksi (206.8 GPa)
- Initial iteration value of parameters $p_i$ = 5.0 sq. in. (32.26 cm$^2$)
- True value of parameters $p_i$ = 3.0 sq in. (19.35 cm$^2$)

As an example, case 5 of Table 1 in the companion paper is now solved with a 1.0\% proportional uniform error applied to both forces and strains and converged in six iterations. The difference between parameter estimates and exact values is known as the percentage error in each parameter estimate and is reported in Table 1. When noise-free simulated measurements were used in the companion paper, the identified parameters were all 3.000; however, when using noisy measurements one should expect contaminated parameter estimates. The percentage error level is highly variable in the estimated cross-sectional areas shown in Table 1.

Two-Dimensional Frame Example with Measurement Noise

\[
\sigma^2 = \frac{I^2}{4} \tag{3}
\]

Eq. (3) is used to generate a set of normal random numbers with a mean of zero and a variance of $I^2/4$.

Having determined these distributions parameterized on $I$, it is possible to generate a set of random numbers based on the applied error value. For the case of forces, these random errors are put into a matrix $\{R_f\}$ of size NDOP × NSF. For strains, a matrix $\{R_s\}$ is formed of size NMS × NSF. Once $\{R_f\}$ and $\{R_s\}$ are formed, they must be applied against either the applied forces, $\{F\}$, or the measured strains, $\{\varepsilon\}$, respectively. There are two ways of applying measurement errors: proportional and absolute. In the case of proportional error, the random numbers are used to represent error percentages, and tend to have a magnitude of less than 1. In this form, the actual errors $\delta$ are calculated by

\[
\delta = [\varepsilon] \odot [R_f] \tag{4}
\]

The $\odot$ = term-by-term (scalar) multiplication. The errors are applied proportionally to simulate contaminated force and strain measurements as

\[
\hat{\{F\}} = \{F\} + [\delta] \tag{5}
\]

\[
\hat{\{\varepsilon\}} = [\varepsilon] + [R_s] \tag{6}
\]

Absolute error can be used to shift the average error from zero. The errors are applied absolutely to simulate contaminated force and strain measurements as

\[
\hat{\{F\}} = \{F\} + [R_f] \tag{4}
\]

\[
\hat{\{\varepsilon\}} = [\varepsilon] + [R_s] \tag{5}
\]

The 2D truss example from Sanayei and Saletnik (1996) is again used and illustrated in Fig. 1. The modulus of elasticity and cross-sectional areas are as follows:

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Noise in the measurement of strains is introduced in the same manner as for forces and was used to compute $\{\varepsilon\}$. For both forces and strains, parameter estimates were all 3.000; however, when using noisy measurements one should expect contaminated parameter estimates. The percentage error level is highly variable in the estimated cross-sectional areas shown in Table 1.

Two-Dimensional Frame Example with Measurement Noise

As an example, case 5 of Table 1 in the companion paper is now solved with a 1.0\% proportional uniform error applied to both forces and strains and converged in six iterations. The difference between parameter estimates and exact values is known as the percentage error in each parameter estimate and is reported in Table 1. When noise-free simulated measurements were used in the companion paper, the identified parameters were all 3.000; however, when using noisy measurements one should expect contaminated parameter estimates. The percentage error level is highly variable in the estimated cross-sectional areas shown in Table 1.

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The structural cross-sectional parameters such as area and moment of inertia may have large differences in magnitude. To eliminate the possibility of one parameter overshadowing the statistics of the others, percentage errors are used as in (7). Then it is possible to summarize across all unknown parameters \( \{\text{obs} \} \) and observations \( \{\text{N OBS} \} \) and formulate the grand mean percentage error as

\[
GM = \frac{1}{\text{NOBS} \cdot \text{N OBS}} \sum_{i=1}^{\text{NOBS}} \sum_{j=1}^{\text{N OBS}} E_{ij}
\]

and the grand standard deviation percentage error as

\[
GSD = \sqrt{\left( \frac{1}{(\text{NOBS} \cdot \text{N OBS}) - 1} \sum_{i=1}^{\text{NOBS}} \sum_{j=1}^{\text{N OBS}} (E_{ij} - GM)^2 \right)}
\]

where \( GM \) and \( GSD \) are system mean and standard deviation, respectively. Any observations that do not converge are not used in computing these statistics. In the same sense that a sample size of 1 is not valid statistically, reducing all these experiments to just two scalar values is not an accurate representation; in particular, the \( GM \) does not show maximums or minimums, but is merely a mean. Monte Carlo simulations were initially performed with 100, 1,000, and 10,000 observations. The \( GM \) consistently reduced when approaching zero. Therefore, it is assumed that the proposed method is an unbiased estimator of the parameters without any formal proof. Assuming an unbiased estimator, the \( GM \) should be zero for a large \( \text{NOBS} \); the expected value of the identified parameters are the true parameters, with an error of zero. Although it is possible to use different levels of measurement error for each applied force and each strain measurement, the input error is selected as a single percentage value representing all possible sources of error. In this sense, it is possible to establish an input-output error relationship such that for a given \( I_0 \), a single value is obtained for \( GM \) and \( GSD \).

### Truss Example, Monte Carlo Analysis

All cases of Table 1 in Sanayei and Saletnik (1996) are reexamined with a 1% random uniform proportional error added to all force and strain measurements. Since Cases 2 and 7 were singular, they were excluded from this study. The number of converged cases in the 1,000 Monte Carlo observations performed as well as the \( GM \) and \( GSD \) are reported and summarized in Table 4.

For most cases all 1,000 observations converged and in some runs a few of the observations failed due to ill-conditioning of the sensitivity matrix. \( GM \) and \( GSD \) are smallest for case 1 and a wide range of values are observed for other cases. This shows that pretest selection of an error-tolerant subset of forces and strains is paramount.

### Frame Example, Monte Carlo Analysis

To demonstrate the input-output error behavior of various subsets of measurements, Monte Carlo simulations were performed for the 2D frame cases reported in Table 3 of the companion paper. Case 6 was not possible to run due to singularity of the sensitivity matrix. In cases 4 and 9, a great majority of the observations did not converge. These results are reported in Table 5.

Except for case 1, none of the Monte Carlo experiments were fully successful; thus, their statistics are not significant. Cases 2–10 show a great variation in \( GM \) and \( GSD \), indicating that the 2D frame structure is not as noise-tolerant as the 2D truss example. In addition, it can be seen that some cases are

### Table 3. 2D Frame Identification % Error, Case 2 with 1% Uniform Proportional Error (Case 2: Applied Forces 3, 5, 9, 12; Measured Strains 1–7)

<table>
<thead>
<tr>
<th>Element</th>
<th>( A )</th>
<th>( A ) Error in ( A )</th>
<th>( l )</th>
<th>( l ) Error in ( l )</th>
<th>( GSD )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75.0</td>
<td>76.944</td>
<td>2.59</td>
<td>2,500.0</td>
<td>964.553</td>
</tr>
<tr>
<td>2</td>
<td>75.0</td>
<td>68.223</td>
<td>9.04</td>
<td>2,500.0</td>
<td>2,418.529</td>
</tr>
<tr>
<td>3</td>
<td>75.0</td>
<td>70.838</td>
<td>5.55</td>
<td>2,500.0</td>
<td>2,463.955</td>
</tr>
<tr>
<td>4</td>
<td>75.0</td>
<td>74.444</td>
<td>0.74</td>
<td>2,500.0</td>
<td>2,475.753</td>
</tr>
<tr>
<td>5</td>
<td>75.0</td>
<td>75.528</td>
<td>0.70</td>
<td>2,500.0</td>
<td>2,508.300</td>
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<tr>
<td>6</td>
<td>150.0</td>
<td>150.271</td>
<td>0.27</td>
<td>2,500.0</td>
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<tr>
<td>7</td>
<td>150.0</td>
<td>148.119</td>
<td>1.25</td>
<td>2,500.0</td>
<td>1,467.954</td>
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</table>

### Table 4. 2D Truss Monte Carlo Analysis, 1% Uniform Proportional Input Error, \( \text{NOBS} = 1,000 \)

<table>
<thead>
<tr>
<th>Case</th>
<th>( \text{NOBS} )</th>
<th>( GM )</th>
<th>( GSD )</th>
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<tr>
<td>1</td>
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<td>0.0072</td>
<td>0.5912</td>
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<tr>
<td>3</td>
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<td>980</td>
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<tr>
<td>8</td>
<td>1,000</td>
<td>0.0250</td>
<td>1.3207</td>
</tr>
<tr>
<td>9</td>
<td>999</td>
<td>1.0394</td>
<td>14.2994</td>
</tr>
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</table>

3. Clearly, there are significant differences between some of the parameter estimates and associated true values in both examples. For this example, it seems that areas are estimated more accurately than the force. However, one sample is not statistically sufficient. Therefore, Monte Carlo analysis is used to study the impact of measurement noise level on parameter estimates.
so highly sensitive to input error that the algorithm cannot converge. Therefore, a pretest study of various subsets of force and strain measurements is required for the selection of a noise-tolerant subset of measurements. Clearly, there is a large variation in output error from case to case in both examples. Next, a technique for the pretest selection of small and noise-tolerant subsets of applied forces and measured strains will be presented.

EXPERIMENT DESIGN FOR NONDESTRUCTIVE TESTING AND PARAMETER ESTIMATION

There is clearly a need for an approach to determine an optimal set of applied forces and measured strains. The number of possible subsets of applied forces and measured strains is combinatorial and leads to an enormous number of permutations. When using NDT data, it is beneficial both in terms of labor and finance to do as little work as possible. However, it is clear that by simply choosing a random subset of forces and strains, errors and ill-conditioned systems can arise. To this end, the algorithm’s behavior in the presence of noisy measurements must be examined. Error sensitivity analysis will be utilized for the experiment design and is expected to lead to a successful parameter estimation with small errors in the parameter estimates.

Error Sensitivity Analysis

The goal of error sensitivity analysis is to determine how a given deterministic error in each particular force or strain measurement affects the error in the parameter estimates. As a technique, it differs greatly from the simulated random measurement error used in Monte Carlo analysis. It also differs from the sensitivity matrix \([S]\) defined in the companion paper.

The error sensitivity matrix, \([S_i]\), represents the change in each identified parameter, \(p_i, i = 1\) to \(N_{UP}\), with respect to a small change in each applied force, \(f_i, i = 1\) to \(N_{SF}\), or each measured strain, \(e_i, i = 1\) to \(N_{MS}\). It is defined as

\[
[S_i] = \begin{bmatrix}
\frac{\partial p_{1}}{\partial f_1} & \frac{\partial p_{1}}{\partial f_2} & \cdots & \frac{\partial p_{1}}{\partial f_{N_{UP}}} \\
\frac{\partial p_{2}}{\partial f_1} & \frac{\partial p_{2}}{\partial f_2} & \cdots & \frac{\partial p_{2}}{\partial f_{N_{UP}}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial p_{N_{UP}}}{\partial f_1} & \frac{\partial p_{N_{UP}}}{\partial f_2} & \cdots & \frac{\partial p_{N_{UP}}}{\partial f_{N_{UP}}}
\end{bmatrix}
\]

(10)

The error sensitivity matrix of (10), \([S_i]\), is numerically evaluated as follows:

1. Choose all the unknown parameters to be identified. The rest are assumed known with a high level of confidence.
2. Select all the available force and strain measurements.
3. Specify a low percentage error (noise) to ensure a linear input-output error relationship.
4. For a force or a strain, apply the predefined error. All other forces and strains are left at their true values.
5. Proceed with an identification, and store the biases of the identified parameters.
6. Repeat steps 4 and 5 for each force and strain, and form \([S_i]\) as defined in (10).

The largest bias of each row is defined as the maximum percentage error \((ME)\) and the largest bias in \([S_i]\) is defined as the maximum percentage error for all parameters \((MPEA)\). \(MPEA\) will be used for the selection of a small subset of noise-tolerant measurements.

### Table 6. 20 Truss, Percentage Errors in Parameter Estimates for 1.0% Error Applied to Error Location

<table>
<thead>
<tr>
<th>Error location (1)</th>
<th>(A_1) (2)</th>
<th>(A_2) (3)</th>
<th>(A_3) (4)</th>
<th>(A_4) (5)</th>
<th>(A_5) (6)</th>
<th>(A_6) (7)</th>
<th>(A_7) (8)</th>
<th>(A_8) (9)</th>
<th>(A_9) (10)</th>
<th>(A_{10}) (11)</th>
<th>MPE (12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_1)</td>
<td>0.041</td>
<td>-0.018</td>
<td>0.536</td>
<td>-0.028</td>
<td>0.682</td>
<td>-0.009</td>
<td>0.001</td>
<td>-0.009</td>
<td>0.003</td>
<td>-0.024</td>
<td>0.682</td>
</tr>
<tr>
<td>(f_2)</td>
<td>0.328</td>
<td>-0.018</td>
<td>-0.048</td>
<td>0.146</td>
<td>0.268</td>
<td>-0.008</td>
<td>0.006</td>
<td>-0.009</td>
<td>0.031</td>
<td>-0.029</td>
<td>0.328</td>
</tr>
<tr>
<td>(f_3)</td>
<td>-0.018</td>
<td>0.041</td>
<td>0.536</td>
<td>-0.028</td>
<td>0.682</td>
<td>0.001</td>
<td>-0.009</td>
<td>-0.024</td>
<td>0.033</td>
<td>0.682</td>
<td></td>
</tr>
<tr>
<td>(f_4)</td>
<td>0.208</td>
<td>0.232</td>
<td>-0.093</td>
<td>0.162</td>
<td>0.043</td>
<td>0.172</td>
<td>0.028</td>
<td>0.529</td>
<td>0.100</td>
<td>0.749</td>
<td>0.749</td>
</tr>
<tr>
<td>(f_5)</td>
<td>0.015</td>
<td>0.219</td>
<td>0.015</td>
<td>0.105</td>
<td>-0.048</td>
<td>0.073</td>
<td>0.845</td>
<td>-0.032</td>
<td>-0.022</td>
<td>0.284</td>
<td>0.845</td>
</tr>
<tr>
<td>(f_6)</td>
<td>0.232</td>
<td>0.208</td>
<td>-0.093</td>
<td>0.162</td>
<td>0.043</td>
<td>0.162</td>
<td>0.028</td>
<td>0.172</td>
<td>0.529</td>
<td>0.749</td>
<td>0.100</td>
</tr>
<tr>
<td>(f_7)</td>
<td>0.015</td>
<td>0.219</td>
<td>0.105</td>
<td>0.073</td>
<td>-0.048</td>
<td>-0.032</td>
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<td>0.022</td>
<td>0.284</td>
<td>-0.142</td>
<td>0.845</td>
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<tr>
<td>(e_1)</td>
<td>-1.007</td>
<td>0.005</td>
<td>0.005</td>
<td>0.010</td>
<td>0.010</td>
<td>0.016</td>
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<td>-0.002</td>
<td>-1.007</td>
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<tr>
<td>(e_2)</td>
<td>0.003</td>
<td>-1.007</td>
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<td>-0.016</td>
<td>0.016</td>
<td>0.010</td>
<td>0.003</td>
<td>-0.011</td>
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<td>-0.002</td>
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<tr>
<td>(e_3)</td>
<td>0.003</td>
<td>-1.021</td>
<td>0.003</td>
<td>0.003</td>
<td>-0.003</td>
<td>0.003</td>
<td>0.003</td>
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<tr>
<td>(e_4)</td>
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<td>-0.881</td>
<td>-0.105</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.881</td>
</tr>
<tr>
<td>(e_5)</td>
<td>0.003</td>
<td>0.011</td>
<td>0.17</td>
<td>-0.881</td>
<td>-0.105</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.881</td>
</tr>
<tr>
<td>(e_6)</td>
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<td>0.000</td>
<td>-0.001</td>
<td>-0.002</td>
<td>0.100</td>
<td>-0.003</td>
<td>0.003</td>
<td>0.006</td>
<td>-0.001</td>
<td>-0.100</td>
</tr>
<tr>
<td>(e_7)</td>
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<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
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<tr>
<td>(e_{10})</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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Truss Example Error Sensitivity Analysis

An example of an error sensitivity analysis is performed using the same 2D truss model. Using case 1 of Table 1 of the companion paper, all forces and strain measurements are available. The identified parameters are cross-sectional areas $A$, with a true value of 3,000 sq in. (19.35 cm$^2$). A 1% proportional uniform error is specified to be applied to the measurements one at a time. The percentage error sensitivities are presented in Table 6. The maximum value per row is indicated by a bold number.

Given a 1% error added to any force or strain measurement one at a time, these are the maximum output errors in the respective identified parameters. These are not guaranteed to be the maximums if errors are present simultaneously; however, they represent only the maximums given one error at a time. They are useful as a representation of error propagation in a multivariate nonlinear system of equations. The largest of the MPE values is the maximum percentage error of all the parameters. The MPE for Table 6, MPE is $-0.055$.

Best-In-Worst-out Method

The data from the error sensitivity analysis is very useful. It is apparent that a random selection of measurements is not adequate. However, using the error sensitivities as a representation of the maximum output error for a given single measurement error, a heuristic strategy is proposed for the pretest selection of subsets of forces and strains, which is called the best-in-worst-out (BIWO) method.

Consider the error sensitivity analysis of the 2D truss reported in Table 6. Take the maximum percentage error for all parameters, MPE, in this case is $-0.055$ for a 1% error.
error added to $\varepsilon$. To determine whether a force or strain can be safely eliminated, rerun the error sensitivity analysis without that measurement and calculate $MPEA$. Continue eliminating each available measurement one at a time, until a table of $MPEA$ values is constructed again. Locate the smallest absolute value of $MPEA$; the system error is least sensitive to eliminating this force or strain measurement. Therefore, it may be eliminated permanently. Repeat this elimination until either the number of measurements, $NM = NMS \times NSF$, becomes less than the number of unknown parameters, $NUP$, or the output error becomes very large. This procedure is summarized as a continuation of the error sensitivity analysis as follows:

1. Perform an error sensitivity analysis using all available measurements (force or strain). Calculate $MPEA$.
2. Temporarily drop one measurement, leaving all others present. Perform error sensitivity analysis and calculate $MPEA$.
3. Repeat step 2 until every force or strain has been dropped once.
4. Determine which measurement has the smallest magnitude $MPEA$ in absolute value. Permanently remove this measurement from testing.
5. Repeat the cycle of steps 7–10 until output error grows unacceptably large, or $NM \leq NUP$.

This heuristic technique is called the BIWO method. A similar method was first presented by Haftka and Adelman (1985) for the shape control of space structures. The best measurements—those that have the highest sensitivity to changes in the parameters and the lowest sensitivity to the errors in the measurements—are retained. The worst measurements—those that do not affect the results significantly—are removed. It is desirable to select a noise-tolerant subset of measurements that is sensitive to changes in the identified parameters. Using the 2D truss and frame, the BIWO method can be seen in use. For each example, a small and noise-tolerant subset of measurements was successfully located. This preselected set of measurement locations can potentially be used for NDT and parameter estimation.

### Truss Example, Best-In-Worst-out Method

The maximum percentage error in all parameter estimates, $MPEA$, for the 2D truss is summarized in Table 7. It begins with all available forces and all available strain measurements. Column 1 of Table 7 represents the force or strain that is dropped for the BIWO method. "None" indicates that all available measurements are present. Case $a$ of Table 7 represents 19 error sensitivity analyses; the first row has 18 measurements, the following rows have 17 (one measurement is dropped at a time). The smallest $MPEA$ value when all measurements are present (row one) is $-1.055\%$. The smallest magnitude $MPEA$ in absolute value for case $a$ is $-1.039\%$, associated with eliminating either $f_2$ or $f_7$. Since the sensitivity analysis is the result of dropping measurements one at a time, only $f_2$ is permanently dropped and the procedure is repeated. In this table, "-" indicates a permanently dropped measured force or strain. If the sensitivity matrix becomes singular due to dropping of a particular measurement, that measurement is readded permanently to the system. These are indicated in the table with an "M". Superscript "a" is used to indicate permanently dropped measurements. The last row of Table 7 indicates the number of measurements, $NM$. The results of 11 computer runs, cases $a$ to $k$, are shown in Table 7.

Based on the first row of Table 7, errors slightly but not drastically increased as further measurements were removed. Case $k$, the smallest subset of measurements, is chosen for the 2D truss and is as follows:

$$f_1, f_2, e_1, e_2, e_3, e_4, e_5, e_6$$

The number of measurements, $NM$, is 12, as compared to the original $NM$ of 80, which represents a significant reduction.

### Frame Example, Best-in-Worst-out Method

The 2D frame also begins with all possible forces and all possible strain measurements. No parameters are known; i.e., $A$ and $I$ are both unknown. Using the same BIWO method as was used for the 2D truss, the maximum percentage error sensitivities are reported in Table 8. Based on the drop analysis, the chosen subset of measurements for the 2D frame is case $l$, as follows:

$$f_4, f_6, f_7, f_{11}, e_1, e_2, e_3, e_4, e_5, e_6$$

This is an example of a situation where the last possible drop run is not the one chosen for further use. Case $l$ has the lowest percentage error of all the runs, that is, $-7.5\%$. Additionally, it is much better-conditioned than case $m$, showing fewer singularities, not only in the drop analysis, but also when doing identifications in the presence of input error. For these reasons, case $l$ is chosen over case $m$ as the potential subset for NDT and parameter estimation.

Overall, this is a very intriguing example. If all forces and strains are measured, there is a 25.8% output error. After eliminating 10 forces and one strain, the maximum output error drops to 7.5%. This is an interesting mathematical phenomenon; it is counterintuitive that output errors would be higher when all possible measurements are available. The BIWO method has produced a set of only 24 measurements with a maximum output error of only 7.5%, case $l$. This is a significant reduction in the number of measurements from 98 to 24, and a significant reduction in the error sensitivity from 25.8% to 7.5%.

### Table 9. 2D Truss Monte Carlo Analysis, Uniform Proportional Input Error, $NOBS = 100$ (Case $k$; Applied Forces 6, 8; Measured Strains 1, 3, 4, 6, 7, 8)

<table>
<thead>
<tr>
<th>$I_0$</th>
<th>$NOBS$</th>
<th>$GM$</th>
<th>$GSD$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>0.00</td>
<td>100</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
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<td>100</td>
<td>0.0001360</td>
<td>0.0099664</td>
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<td>100</td>
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<td>10.7380270</td>
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<tr>
<td>50.00</td>
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<td>26.5965290</td>
<td>100.0809000</td>
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</table>

### Table 10. 2D Frame Monte Carlo Analysis, Uniform Proportional Input Error, $NOBS = 100$ (Case $l$; Applied Forces 4, 6, 7, 11; Measured Strains 1, 2, 4, 5, 6, 7)

<table>
<thead>
<tr>
<th>$I_0$</th>
<th>$NOBS$</th>
<th>$GM$</th>
<th>$GSD$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
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<td>0.0</td>
</tr>
<tr>
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<td>100</td>
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<td>0.0577509</td>
</tr>
<tr>
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<tr>
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<td>100</td>
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<td>1.7340924</td>
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<tr>
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<td>11.2224840</td>
<td>72.0969300</td>
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Input-Output Error Relationships

At this point, Monte Carlo analysis is used to determine an input-output error relationship \((I_e - O_e)\) for the final choices of measurements at several levels of \(I_e\). For each \(I_e\), uniformly distributed random errors are added proportionally to all measurements. Monte Carlo analysis is performed using 100 observations and the parameters are identified for each observation. By performing a statistical analysis on the resulting output error observations, the GM and GSD values are tabulated against \(I_e\), varying from zero to 50% for the truss example and from zero to 10% for the frame example. The input-output error relation for the truss and frame examples using the selected cases from the BIWO method are reported in Tables 9 and 10, respectively.

To examine the input-output error relationships, it is desired to form a family of line graphs based on several cases of the BIWO method reported in Tables 7 and 8 by plotting the GSD values against \(I_e\) values. A family of line graphs is shown in Fig. 3 for the truss example, or in Fig. 4 for the frame example. Figs. 3 and 4 are used to estimate the output error given an input error. These graphs can be used to determine the allowable \(I_e\) by limiting output error in each example for the experiment design. The measurement noise tolerance is expected to vary from structure to structure based on the number of measurements, location of measurements, and topology of the structure. If the required input error cannot be achieved, then another case with more measurements from the BIWO method results can be used. It is clear that the frame example does not tolerate measurement noise levels as high as those of the truss example.

Comparison of Experiment Design with Other Selections

The final measurement selection for the truss example using the proposed BIWO method, case \(k\), is compared with randomly selected cases of Table 4 using a 1% uniform proportional measurement error. For comparison, GSD of cases that identified all 10 parameters in Table 4 and case \(k\) of Table 7 are plotted using a bar chart and are shown in Fig. 5. For cases with a large number of measurements such as cases 1, 4, and 5, the GSD is fairly low. However, when more measurements are eliminated such as cases 6, 9, and 10, the GSD increases drastically. Using a small number of measurements, case \(k\) generates a smaller output error of about 5% compared to cases with a similar number of measurements, such as cases 6, 9, and 10.

For the frame example, the selected cases \(l\) and \(m\) using the BIWO method of Table 8 produced GSD much smaller than any combination of measurements in Table 5, identifying all 14 parameters. These GSDs are shown in Fig. 6. Selected case \(l\) is performing even better than case 1 in which all possible measurements are used. Although it is counterintuitive, more measurements does not always lead to more accurate parameter estimates. This phenomenon is due to the fact that all measurements include some degree of error, and for certain types of structures they start to add up and cause high errors in the parameter estimates.

Using the BIWO method, a small and noise-tolerant subset of measurements is established for the truss example and for the frame example. These designed experiments can potentially be used for NDT and parameter estimation for damage assessment. After the initial simulation for the experiment design, only one NDT and one parameter estimation computer run is required for each structure.

In actual nondestructive testing there are always stations that are impossible to apply forces to and stations that are difficult or impossible to measure the responses for. In addition, for frame structures it is difficult to apply a torque to any stations and, in most cases, applied forces are gravity loads. Therefore, the initial set of measurements is always reduced to a smaller subset of possible measurements, allowing selection of the best subset out of the remaining stations. These considerations will reduce the size of the combinatorial problem to be solved by the BIWO method.
To perform any parameter estimation, a nondestructive test must first be designed. The purpose of the experiment design is to determine a small and noise-tolerant subset of forces and strains to be used for NDT. It must be decided how many force and strain measurements are required and the most appropriate locations for these measurements. In addition, the acceptable precision of the sensors and transducers must be determined based on the acceptable error in the parameter estimates.

This study is based on the parameter estimation technique formulated in a companion paper, using subsets of applied static forces and subsets of strain measurements. Since the measure of success for the experiment design using NDT data is the level of error in the identified parameters, first an input-output error analysis was successfully simulated for the proposed method. This study showed that arbitrary or even judgmental selection of subsets of applied forces and subsets of strain measurements can potentially lead to error-sensitive parameter estimation systems. This behavior results from different topologies caused by the nonlinear behavior of the error function that is minimized to estimate the parameters. Therefore, an error sensitivity analysis was performed to study and compare various subsets of measurements and their impact on the parameter estimates.

To this end, a series of error sensitivity analyses were performed. By simulating measurements with a given applied error, the noncritical measurements are located and eliminated. The repetition of this task continues until either the output error becomes too great, or no more measurements can be eliminated due to singularities in the sensitivity matrix. This heuristic method is called the best-in-worst-out method and is used for the selection of a subset of applied forces and a subset of strain measurement locations, leading to a noise-tolerant parameter estimation system. It is shown that it is possible for
output error levels to decrease as measurements are eliminated; the use of all available measurements is not always the better choice. However, certain measurements are critical to a successful evaluation and should not be eliminated. There is no proof that the selected subset of measurements will be optimal; however, it is expected to be near-optimal, leading to a promising set of measurements for NDT and parameter estimation for damage assessment.

Once an appropriate set of forces and strains was selected, a Monte Carlo analysis was performed selecting a measurement error with a 95% confidence interval (input error). By performing a statistical analysis on the errors in the identified parameters (output error), an estimate of the input-output error behavior was established. This was followed by a series of Monte Carlo analyses for several different input errors to determine and plot the input-output error behavior of the selected experiment. Using this plot, an interval of acceptable measurement error leading to parameter estimates with errors within the predefined acceptable range was found, and the experiment design was successfully accomplished.

The experiment design is the pretest simulation process that leads to the selection of a noise-tolerant subset of measured forces and strains to be used for NDT. Then, it is possible to perform only a single actual NDT of significantly fewer measurements for data acquisition and parameter estimation. Using this data, there is a high possibility of a successful parameter estimation for the predesigned experiment. Future work includes large-scale testing of structures and development of techniques to examine parameter estimates to detect any structural damage. Additionally, this work can potentially be used for on-line health monitoring and damage assessment of structures.

APPENDIX. REFERENCES


