On approximate symmetries of the elastic properties and elliptic orthotropy

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Abstract

The theory of anisotropic elasticity was originally motivated by applications to crystals, where geometric symmetries hold with high precision. In contrast, symmetries of the effective elastic responses of heterogeneous materials are usually approximate due to various imperfections of microgeometry. A related issue is that available data on the elastic constants may be incomplete or imprecise; it may be appropriate to select the highest possible elastic symmetry that fits the data reasonably well. Some of these problems have been discussed in literature in the context of specific applications, primarily in geomechanics. The present work provides a systematic discussion of the related issues, illustrated by examples on the effective elastic properties of heterogeneous materials. We also discuss a special type of orthotropy typical for a variety of heterogeneous materials—elliptic orthotropy—when the fourth-rank tensor of elastic constants can be represented in terms of a certain symmetric second-rank tensor. This representation leads, in particular, to reduced number of independent elastic constants.

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1. Introduction

The theory of anisotropic elasticity was originally motivated by applications to crystals, where geometric symmetries hold with high precision. Due to that, symmetry of elastic response (which will be called hereafter elastic symmetry) has been traditionally considered as exact ones. Under the assumption that elastic symmetries are exact, various representations of anisotropic fourth-rank tensors have been developed (see works of Walpole [28], Cowin and Mehrabadi [5,6], Ting [25], Theocaris and Sokolis [23]).

In contrast, symmetries of the effective properties of heterogeneous materials, have, typically, approximate character, due to various “irregularities” of microstructures (irregular shapes or fuzzy orientation distribution of inhomogeneities). Examples of such microstructures are discussed in Section 4. Aside from the actual elastic constants, the available information on them may be incomplete or imprecise. It may be
appropriate, then, to simply select the highest possible elastic symmetry (orthotropy, transverse isotropy or even isotropy) that fits the data reasonably well. Fractured rocks and trabecular bone tissue, where wavespeed data providing information on the effective elastic constants are “noisy”, serve as examples. Problems of this kind constitute the primary motivation for the present work.

A related issue is that elastic symmetries – defined as exact ones – may experience discontinuous changes when the elastic constants change only incrementally. An example is given by the emergence of weak anisotropy due to slightly non-random orientations of non-spherical inhomogeneities. Such symmetry “jumps” appear undesirable since they are not associated with any transition points of importance.

Our second subject is the special type of orthotropy – *elliptic orthotropy* – when the tensor of elastic constants can be represented in terms of certain symmetric second-rank tensor \(\omega\). It holds, as an approximation, for the effective elastic properties of broad classes of materials with inhomogeneities (cracks, pores, inclusions) and has important implications. First, the orthotropic symmetry holds for any orientation distribution of inhomogeneities (even if, geometrically, the distribution does not have this symmetry). Second, the orthotropy is of a special kind: there are only six independent constants (this number is further reduced, to only four, in the case of crack-induced anisotropy). Third, it constitute one of the basic assumptions leading to elasticity–conductivity connections [21], provided tensor \(\omega\) is explicitly expressed in terms of relevant microstructural parameters.

In materials science, approximate symmetries are usually treated in an intuitive way, as being present if deviations from them seem reasonably small. This, however, may be insufficient if the observations are to be quantified. An example is given by interpretations of anisotropic wavespeed patterns in geomaterials. Quantitative approach to approximate symmetries was pioneered by Fedorov [8] who derived the best isotropic approximation of a given elastic symmetry (his results were rewritten in a different form by Cavallini [3]). In geophysical applications, his approach was extended to the best *anisotropic* approximation of a given set of elastic constants by Arts et al. [1,2] and Helbig [12]. One of the problems is finding the optimal orientation of anisotropy axes of the approximating medium; guidance in this respect is provided by results of Cowin and Mehrabadi [5] for the orientation of the *exact* symmetries in cases the latter exist but elastic constants are given in an arbitrary coordinate system. In the context of geophysics, a different approach was taken by Thomsen [24] who introduced three parameters characterizing the deviation of wave propagation patterns in transversely isotropic rocks from the ones in the isotropic material, and treated them as indicators of the deviation from isotropy.

The present work provides a systematic overview of these issues and discusses their application to the effective elastic properties of anisotropic heterogeneous materials. Under the usual assumption that ergodic hypothesis holds, we consider the effective elastic properties as averages over representative volume. Section 2 aims at clarifying the basic issues and presenting relevant results in the simplest form possible. Section 3 focuses on the special kind of orthotropy that holds, approximately, for a broad class of materials with inhomogeneities – the elliptic orthotropy. Section 4 provides examples related to materials science applications.

2. The concept of approximate elastic symmetry

2.1. The basic definition

The usual definitions of elastic symmetry assume that symmetry elements are either present or not [19]. According to such definitions, symmetries simply do not exist if they are approximate. Besides being overly restrictive for materials science applications, they lead to an undesirable feature: small changes in elastic constants may produce discontinuous changes in elastic symmetries. Simply setting a tolerance threshold for symmetry violations does not eliminate the “jumps” but shifts them to the threshold point.

We suggest the following definition of the elastic symmetry:

*Any element of elastic symmetry is always present, with certain accuracy that is measured by appropriately chosen norm*

This definition eliminates symmetry “jumps”: continuous changes in elastic constants produce continuous changes in the accuracy. For example, “weak anisotropy” means that the error of the statement that the material is isotropic is small. The definition requires the choice of norm that measures the accuracy.
2.2. The Euclidean norm and the best isotropic approximation of elastic anisotropies

The simplest norm is the Euclidean one. It defines the difference between two compliance tensors, \( \mathbf{S} \) and \( \overline{\mathbf{S}} \) (or stiffness tensors, \( \mathbf{C} \) and \( \overline{\mathbf{C}} \)) as

\[
||\mathbf{S} - \overline{\mathbf{S}}|| = \sqrt{\sum (S_{ijkl} - \overline{S}_{ijkl})(S_{ijkl} - \overline{S}_{ijkl})}
\]

so that the error of the approximation of \( \mathbf{S} \) by \( \overline{\mathbf{S}} \) is

\[
\delta = \sqrt{\left(\frac{(S_{ijkl} - \overline{S}_{ijkl})(S_{ijkl} - \overline{S}_{ijkl})}{S_{pqrs}S_{pqrs}}\right)}
\]

(the usual summation convention is assumed). This norm was used by Fedorov [8] who was, probably, the first to address the problem of approximate elastic symmetries in quantitative terms. He used it in calculations of the best isotropic approximation of elastic anisotropies. It was extended to the best anisotropic approximation of available data, by a tensor of elastic constants possessing a given symmetry, by Arts et al. [1,2] and Helbig [12], in the context of geophysical applications.

We overview the best isotropic approximation of a general anisotropy, rephrasing the result of Fedorov [8] in a somewhat different form. Representing the isotropic approximation of tensor \( \lambda_{ijkl} \) (that may represent anisotropic compliances \( S_{ijkl} \) or stiffnesses \( C_{ijkl} \)) as

\[
c_d\delta_{ikl} + a(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\]

the constants \( a, c \) providing the best Euclidean fit of \( \lambda_{ijkl} \) are given by

\[
a = (3\lambda_{ik} - \lambda_{ikk})/30, \quad c = (2\lambda_{ik} - \lambda_{ikk})/15
\]

Taking \( \lambda_{ijkl} \) as compliances \( S_{ijkl} \), the best-fit shear and bulk moduli are \( G^* = 1/(4a) \) and \( K^* = 1/(9c + 6a) \).

Remark. The isotropic approximation contains no references to the non-orthotropic components of \( \lambda_{ijkl} \) (such as \( \lambda_{1112} \)) – their values do not affect constants \( a \) and \( c \).

We further specify this result for the case when \( \lambda_{ijkl} \) are orthotropic compliances expressed in Young’s and shear moduli and Poisson’s ratios (\( S_{1111} = 1/E_1, \ S_{1212} = 1/2G_{12}, \ S_{1122} = -v_{12}/E_1 \), etc.). The best isotropic fit \( G^* \) and \( K^* \) is given by

\[
15G^*_1 = 4[(1 + v_{12})E_1^{-1} + (1 + v_{23})E_2^{-1} + (1 + v_{31})E_3^{-1}] + 3(G_{12}^{-1} + G_{23}^{-1} + G_{31}^{-1})
\]

\[
K^*_1 = (1 - 2v_{12})E_1^{-1} + (1 - 2v_{23})E_2^{-1} + (1 - 2v_{31})E_3^{-1}
\]

Note that the result for \( K^* \) is simple and has expected structure, for example, it does not depend on the shear moduli; whereas the result for \( G^* \) is not intuitively obvious and incorporates all the orthotropic constants. For the cubic symmetry (all \( E_i = E \), all \( G_{ij} = G \) and all \( v_{ij} = v \) in the coordinate system aligned with the cube axes), the best isotropic fit is

\[
5G^*_1 = 4(1 + v)/E + 3/G, \quad K^*_1 = E/[3(1 - 2v)]
\]

2.3. Best-fit orthotropic and transversely isotropic approximations

We seek to approximate a given compliance matrix \( S_{ijkl} \) by the best-fit matrix \( \overline{S}_{ijkl} \) that has ‘\( a \ priori \)’ specified anisotropy. The problems to be addressed are:

- specifying the type of symmetry of \( \overline{S}_{ijkl} \) (orthotropy, transverse isotropy, isotropy) that, based on the information available (microstructure, wavespeed data) is expected to be a reasonable approximation;
- identifying the best-fit orientation of the orthotropic axes of \( \overline{S}_{ijkl} \);
- Finding the best-fit values of \( \overline{S}_{ijkl} \).

We focus on the orthotropic or transversely isotropic approximations \( \overline{S}_{ijkl} \) that are of main interest for the materials science applications.
The approximate orthotropic symmetry holds, if compliance tensor \( S \) can be approximated by certain orthotropic tensor \( \hat{S} \) with error not exceeding the given value \( \delta \) (as measured by (2.2)). The orientation of the orthotropy axes of \( \hat{S} \) can be found using results of Cowin and Mehrabadi [5] for the presence of exact symmetries if the information on elastic constants is given in an arbitrary coordinate system (Appendix). If \( S \) is exactly orthotropic, the principal axes of two second-rank tensors, \( S_{ijk} \) and \( S_{ikjk} \), are the same and they coincide with the orthotropy axes. Hence, if the mentioned principal axes do coincide, the axes of orthotropy – provided the latter exists – are necessarily these axes; transforming the matrix \( S_{ijkl} \) to them, we verify whether it has the orthotropic form. To test for approximate orthotropy, we first find the principal axes of \( S_{ijk} \) and \( S_{ikjk} \) characterized by two sets of directional cosines. We then select the median, between the two, coordinate system \( \vec{x}_1, \vec{x}_2, \vec{x}_3 \) (characterized by averaged directional cosines) and transform components of tensor \( S_{ijkl} \) to this coordinate system.

**Remark.** If orientations of the principal axes of \( S_{ijk} \) and \( S_{ikjk} \) are substantially different, the procedure remains valid but the accuracy of the orthotropic approximation may be low.

Expressing the compliances in this system, \( S_{ijkl} \), we focus on finding the matrix \( \hat{S}_{ijkl}^{\text{ort}} \) that provides the best orthotropic fit of \( S_{ijkl} \). Of 21 compliances \( S_{ijkl} \), we distinguish nine “orthotropic” components \( S_{ijkl}^\prime \) (subscripts 1111, 2222, 3333, 1122, 2233, 3311, 1212, 2323, 3131) from twelve “non-orthotropic” ones, \( S_{ijkl}^\prime \) (subscripts 1112, etc.). The best orthotropic fit \( \hat{S}_{ijkl}^{\text{ort}} \) is obtained simply by equating them to the corresponding orthotropic components \( S_{ijkl}^\prime \) and setting \( S_{ijkl}^\prime = 0 \). This follows from minimization of the Euclidean norm of the difference \( S_{ijkl} - \hat{S}_{ijkl}^{\text{ort}} \). The error of this approximation is

\[
\delta = \sqrt{\hat{S}_{ijkl}^{\prime\prime} \hat{S}_{ijkl}^\prime / \hat{S}_{ijkl} S_{ijkl}} \tag{2.7}
\]

**Remark.** Finding the best-fit orthotropic sti\(nesses \) \( \hat{C}_{ijkl}^{\text{ort}} \) of the given \( C_{ijkl} \) involves the analogous procedure. Note that \( \hat{S}_{ijkl}^{\text{ort}} \) and \( \hat{S}_{ijkl}^{\text{ort}^\prime} \) providing the best orthotropic fits of \( S_{ijkl} \) and its inverse, \( C_{ijkl} \), are not necessarily inverse to one another in the exact sense.

The same procedure can be applied to finding the best transversely isotropic fit \( \hat{S}_{ijkl}^{\text{TI}} \). If \( x_3 \) is the axis of transverse isotropy, then

\[
\hat{S}_{1111}^{\text{TI}} = \frac{3(\hat{S}_{1111}^\prime + \hat{S}_{2222}^\prime) + 2\hat{S}_{1122}^\prime + 4\hat{S}_{1212}^\prime}{8}; \quad \hat{S}_{3333}^{\text{TI}} = \hat{S}_{3333}^\prime
\]

\[
\hat{S}_{1122}^{\text{TI}} = \frac{\hat{S}_{1111}^\prime + \hat{S}_{2222}^\prime + 6\hat{S}_{1122}^\prime - 4\hat{S}_{1212}^\prime}{8}; \quad \hat{S}_{1212}^{\text{TI}} = \frac{1}{2}(\hat{S}_{1111}^{\text{TI}} - \hat{S}_{1122}^{\text{TI}});
\]

\[
\hat{S}_{1133}^{\text{TI}} = \frac{1}{2}(\hat{S}_{1133}^\prime + \hat{S}_{2233}^\prime); \quad \hat{S}_{1313}^{\text{TI}} = \frac{1}{2}(\hat{S}_{1313}^\prime + \hat{S}_{2333}^\prime)
\]

Formula (2.8) are similar to the ones of Arts et al. [2]; the latter seem to contain a misprint. The error of this approximation is given by the formula similar to (2.7):

\[
\delta = \left[ \sum (\hat{S}_{ijkl}^{\text{TI}} - S_{ijkl})(\hat{S}_{ijkl}^{\text{TI}} - S_{ijkl}) / \sum S_{pqr}s_{pqr} \right]^{1/2} \tag{2.9}
\]

**Remark.** We mention recent paper of Dellinger [7] where a computational algorithm was published for finding the best-fit transversely isotropic approximations.

### 2.4. Measures other than Euclidean norm

The Euclidean norm has been used in all the cited works, as well as the present work, since it is most convenient from the computational viewpoint. However, from the physical point of view, the choice of the norm should be dictated by needs of the specific applications, and the Euclidean norm may not always be the most appropriate choice.
An example of a different norm is provided by the elastic potential, for instance, the one in stresses, \( f(\sigma_{ij}) \): the two compliance tensors, \( S_{ijkl} \) and \( \overline{S}_{ijkl} \), are sufficiently close if \( f(\sigma_{ij}) \) is sufficiently well approximated by \( \bar{f}(\sigma_{ij}) \), i.e.

\[
|f(\sigma_{ij}) - \bar{f}(\sigma_{ij})| = |(S_{ijkl} - \overline{S}_{ijkl})\sigma_{ij}\sigma_{kl}| \ll (\sigma_{ij})
\]

(2.10)

for all stress states \( \sigma_{ij} \) [15]. Although it is equivalent to the Euclidean one in the asymptotic sense (if one of them tends to zero, the other one does, too), the two norms do not coincide: the norm (2.10) is more sensitive to a mismatch in just one constant.

We mention yet another measure, often used in geophysics in the case of transverse isotropy (\( x_3 \) being its axis). The degree of anisotropy is estimated by the three dimensionless parameters introduced by Thomsen’s [24]:

\[
e = \frac{C_{1111} - C_{3333}}{2C_{3333}}, \quad \gamma = \frac{C_{1212} - C_{2323}}{2C_{2323}}, \quad \delta = \frac{(C_{1133} + C_{2333})^2 - (C_{3333} - C_{2333})^2}{2C_{3333}(C_{3333} - C_{2333})}
\]

(2.11)

(In the case of isotropy, \( e = \gamma = \delta = 0 \); parameter \( e - \delta \) is called anelasticity.)

Thomsen’s parameters do not constitute a norm and hence cannot be used as a measure of closeness of the two tensors of elastic constants – the transversely isotropic one and the one of the approximating isotropic medium (for example, they do not allow one to estimate which of the two approximating tensors is closer to the original one). Nevertheless, they capture those combinations of \( e \), \( \gamma \), \( \delta \), and \( \lambda \) that control typical seismic signatures and hence are useful for processing them. These parameters were generalized to the orthotropic and monoclinic media in works of Grechka and Tsvankin [9] and Grechka et al. [10].

3. Elliptic orthotropy

We call the orthotropy elliptic if the fourth-rank tensor of elastic compliances \( S \) can be represented as tensorially linear function of certain symmetric second-rank tensor \( \omega \). This case is important because such representations hold, as an approximation, for the effective elasticity of a broad class of materials with inhomogeneities (inclusions, pores, cracks). For them, \( \omega \) is the parameter of concentration of inhomogeneities (that accounts for their orientations and shapes). The effective compliances of such materials can be represented as a sum

\[
S_{ijkl} = S_{ijkl}^0 + \Delta S_{ijkl}
\]

(3.1)

where “0” refers to the bulk material and the second term is the change due to inhomogeneities. If the bulk material is isotropic,

\[
S_{ijkl}^0 = \frac{1 + v_0}{2E_0} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{v_0}{E_0} \delta_{im}\delta_{jn}\delta_{ks}\delta_{ls}
\]

(3.2)

then tensor \( \Delta S \) must be the isotropic function of \( \omega \) (subjecting \( \omega \) to an orthogonal transformation, for example, rotation over certain angle, would rotate \( \Delta S \) over the same angle). This leads to the following structure

\[
\Delta S_{ijkl} = A_1(\omega_{ij}\delta_{jk} + \omega_{jk}\delta_{ij} + \omega_{il}\delta_{jk} + \omega_{lk}\delta_{il}) + A_2(\delta_{ij}\omega_{kl} + \delta_{kl}\omega_{ij})
\]

(3.3)

where coefficients \( A_{1,2} \) are functions of the elastic constants of the matrix and the inhomogeneities; they also depend on shapes of inhomogeneities and therefore cannot be treated as material constants (see the discussion to follow). Tensor \( \omega \) is a parameter of concentration of inhomogeneities (in the case of cracks or spheroidal inhomogeneities, it is, respectively, the crack density tensor (4.2) or a second-rank tensor that explicitly reflects the distribution over orientations and aspect ratios).

Equivalently, this representation can be given in terms of elastic potentials – for example, the one in stresses, \( f(\sigma, \omega) = f_0(\sigma) + \Delta f(\sigma, \omega) \) so that \( S_{ijkl}\sigma_{kl} = \partial f / \partial \sigma_{ij} \). If the bulk material is isotropic, \( \Delta f \) is expressed in terms of simultaneous invariants of \( \sigma \) and \( \omega \); requiring that it is quadratic in stresses and tensorially linear in \( \omega \) leads to the structure equivalent to (3.2) and (3.3):
Micromechanical analyses identify the following cases when these representations hold with satisfactory accuracy:

- Cracks of arbitrary orientation distributions [14]. In this case, \( \omega \) is the crack density tensor \( \alpha \) that is defined, for the circular cracks, by (4.2) in the text to follow (and by its “effective” for flat cracks of irregular shapes provided the irregularities are random [11]). Note that, in the case of cracks, \( A_2 = 0 \) reflecting the fact that flat cracks do not generate additional strains when loaded in directions parallel to them.
- Arbitrary mixture of spheroidal inhomogeneities of diverse aspect ratios and orientations [21]. In this case, generally, \( A_2 \neq 0 \) reflecting the fact that, in contrast with cracks, pores and inclusions do generate additional strains when loaded in directions parallel to them.

In cases of an anisotropic background, such representations do not follow from any general considerations (since \( \Delta f(\sigma, \omega) \) does not have to be the isotropic function of its arguments and hence may not reduce to their simultaneous invariants). However, micromechanical analyses show that, in the 2D cases of the orthotropic background with cracks and elliptical holes of diverse orientations and aspect ratios, similar representations do hold, with appropriate adjustment in (3.2) or in \( f_0 \) [26].

**Remark 1.** Generally, the requirement of isotropy of the function \( \Delta f(\sigma, \omega) \) leads to representations involving a number of extra terms (in addition to the two terms in (3.4)), that are tensorially non-linear in \( \omega \) (such as \( (\sigma \cdot \sigma)(\omega \cdot \omega), (\sigma; \omega)^2 \), etc.). The fact that \( \Delta f \) contains only two terms (and, moreover, only one term in the case of cracks, since \( A_2 = 0 \) in this case) follows from specific micromechanical analyses that involve, in particular, explicit identification of \( \omega \) in terms of relevant microstructural features.

**Remark 2.** As noted above, coefficients \( A_{1,2} \) depend on the shapes of inhomogeneities and cannot, therefore, be treated as material constants (for example, \( A_2 = 0 \) for cracks, whereas for pores and inclusions \( A_2 \neq 0 \)).

Micromechanical analyses also demonstrate that these representations are not always possible. For example, they may not hold for cracks and pores filled with compressible fluid, for the reason that \( \omega \) must be supplemented by a fourth-rank tensor; interestingly, the latter gives rise to only one extra term [22]. The validity of representations in terms of \( \omega \) is also unclear for inhomogeneities of complex shapes.

Importantly, tensor \( \omega \), as well as factors \( A_{1,2} \) are explicitly expressed in terms of microstructural parameters. This distinguishes representations (3.3) and (3.4) from the “fabric” ones that look similar but are postulated, rather than derived, with \( \omega \) and \( A \)-factors being uncertain quantities that are not explicitly related to the microstructure and play the role of fitting parameters. This leads to much larger number of terms (so that the number of fitting parameters may be quite large) and the possibility to omit them remains unclear without the micromechanical analyses (see Kachanov and Sevostianov [16] for a discussion in detail).

Representation (3.3) gives compliances as linear functions of \( \omega \). Its inversion produces stiffnesses that are non-linear in \( \omega \). Conversely, one may start with representation of stiffnesses linear in \( \omega \); the inversion would yield non-linearity of compliances. In the asymptotics of small \( \omega \), the two representations are equivalent, but they have different ranges of applicability as the concentration \( \omega \) increases. For cracks, that are displacement discontinuities and thus are naturally treated as sources of additional strains, as well as for pores, the representations linear in compliances are more appropriate, in the sense that they retain accuracy at substantially higher concentrations (as demonstrated by computational studies for cracks [11]); for rigid inclusions, representations linear in stiffnesses are more appropriate. In intermediate cases of inhomogeneities of finite stiffness, there seems to be no general guidance for choosing one of the formulations over the other.

Representations (3.3) and (3.4) have far reaching implications, as follows.

(A) They imply the orthotropic symmetry (coaxial with the principal axes of \( \omega \)) that holds for any orientation distribution of inhomogeneities, even if the geometric pattern of the distribution does not have this symmetry. Moreover, the orthotropy has rather special, “elliptic” character, with the number of independent
constants reduced from nine to six. In the principal axes of \( \omega \), this reduction is expressed by the following three relations:

\[
\begin{align*}
 f_1 &= 4S_{1212} - S_{1111} - S_{2222} + 2S_{1122} = 0 \\
 f_2 &= 4S_{2323} - S_{2222} - S_{3333} + 2S_{2323} = 0 \\
 f_3 &= 4S_{3131} - S_{3333} - S_{1111} + 2S_{3331} = 0
\end{align*}
\] (3.5a)

In terms of Young’s and shear moduli and Poisson’s ratios \( (E_1 = 1/S_{1111}, \ G_{12} = 1/(4S_{1212}), \ v_{12}/E_1 = v_{21}/E_2 = -S_{1122}, \text{ etc.}) \), these relations state that the shear moduli are not independent:

\[
\frac{1}{G_{ij}} = \frac{1 + \nu_{ij}}{E_i} + \frac{1 + \nu_{ji}}{E_j}, \quad ij = 12, 23, 31
\] (3.5b)

Note that these reductions hold in spite of the fact that the elliptic character of orthotropy does not imply any additional symmetry elements. For transverse isotropy of the elliptic type, only one of relations (3.5) is non-trivial, thus reducing the number of independent constants from five to four.

Relations (3.5) were proposed, as a hypothesis, simply to reduce the number of independent constants, by Cauchy (according to Love [18]) or by Saint-Venant (according to Lekhnitsky [17]). We find it interesting that they emerge in the context of effective elasticity of certain class of heterogeneous materials.

For crack-induced anisotropy, the number of independent constants is further reduced, to only four [14]. In this case, the following two additional relations hold in the principal axes of the crack density tensor (4.2):

\[
S_{1122} = S_{2233} = S_{3311} (= S^0_{1122} = -v_0/E_0)
\] (3.6)

For the transversely isotropic crack distributions (for example, parallel cracks, or cracks with normals randomly oriented in a plane, both examples may include random orientation scatter), only one of relations (3.6) is non-trivial, reducing the number of independent constants to three.

**Remark.** In contrast with (3.5), relations (3.6) hold only for flat cracks or narrow, crack-like pores. Hence, these relations can be used as a test: if they hold, then the inhomogeneities have crack-like shapes.

**(B) Representation (3.3)** leads to explicit elasticity–conductivity cross-property connections for materials with inhomogeneities, due to the fact that the conductivity changes caused by inhomogeneities can also be expressed in terms of \( \omega \) [21].

**Test for the elliptic orthotropy (EO)** consists in verifying whether relations (3.5) hold. The most straightforward approach to deriving this test is to impose three constraints (3.3) on the general orthotropy test, using the Lagrange multipliers technique. This yields

\[
\begin{align*}
S^0_{1111} &= S_{1111} + (\lambda_1 + \lambda_2)/2, \quad S^0_{2222} = S_{2222} + (\lambda_1 + \lambda_3)/2, \quad S^0_{3333} = S_{3333} + (\lambda_2 + \lambda_3)/2, \\
S^0_{1122} &= S_{1122} - \lambda_1/2, \quad S^0_{1133} = S_{1133} - \lambda_2/2, \quad S^0_{2233} = S_{2233} - \lambda_3/2, \\
S^0_{1212} &= S_{1212} - \lambda_1/2, \quad S^0_{1313} = S_{1313} - \lambda_2/2, \quad S^0_{2323} = S_{2323} - \lambda_3/2
\end{align*}
\] (3.7)

where Lagrange’s multipliers are as follows:

\[
\lambda_1 = \frac{9f_1 - f_2 - f_3}{70}; \quad \lambda_2 = \frac{9f_2 - f_1 - f_3}{70}; \quad \lambda_3 = \frac{9f_3 - f_1 - f_2}{70}
\] (3.8)

and \( f_i \) are given by (3.5a). The error of this approximation measured by Euclidean norm, is

\[
\delta = \sqrt{(S^0_{ijkl} - S^0^E_{ijkl})^2 / S^0_{ijkl}S^0_{ijkl}}
\] (3.9)

4. Examples related to the effective elastic properties of heterogeneous materials

We illustrate the procedures by several examples related to the effective elastic properties of heterogeneous materials. The first of them deals with idealized configuration of two families of circular cracks. The other three deal with real experimental data for different heterogeneous materials – plasma-sprayed coating, cortical bone and carbon fiber reinforced composite.
4.1. Approximate effective orthotropy of a cracked solid

In the non-interaction approximation – that, as demonstrated by computational studies of Grechka and Kachanov [11], remains accurate up to crack densities of at least 0.14 – the change in the tensor of elastic compliances due to multiple circular cracks of an arbitrary orientation distribution is given by Kachanov [13]:

$$\Delta S_{ijkl} = \frac{32(1-v_0^2)}{3(2-v_0)E_0} \left[ \frac{1}{4} (\chi_{ik} \delta_{jl} + \chi_{ij} \delta_{jk} + \chi_{jl} \delta_{ik} + \chi_{jk} \delta_{il}) - \frac{v_0}{2} \beta_{ijkl} \right]$$  (4.1)

where

$$\chi = (1/V) \sum (a^3 nn)^i$$  (4.2)

is a symmetric second-rank crack density tensor and

$$\beta = (1/V) \sum (a^3 mnn)^i$$  (4.3)

is a fourth-rank tensor ($n^{(i)}$ and $d^{(i)}$ are unit normal to $i$-th crack and its radius and $V$ is the representative volume).

Since $\chi$ is a symmetric second-rank tensor, the term in the first parenthesis of (4.1) possesses the elliptic orthotropy for any orientation distribution of cracks, with the orthotropy axes coinciding with the principal axes of $\chi$. The $\beta$-term may produce deviations from orthotropy. However, this effect is generally small, due to (1) a relatively small factor $v_0/2$ at the $\beta$-term and (2) the fact that a substantial part of the $\beta$-term actually contributes to orthotropy. We illustrate it on the following example.

We consider two families of parallel circular cracks inclined at 30° to each other, with partial crack densities $\rho/3$ and $2\rho/3$. Then $\chi = (2\rho/3)n_1n_1 + (\rho/3)n_2n_2$ and $\beta = (2\rho/3)n_1n_1n_1 + (\rho/3)n_2n_2n_2$. The first eigenvector, $e_1$, of $\chi$ is rotated 9.6° from $n_1$ towards $n_2$, the second one, $e_2$, is normal to $e_1$; the third one is along the $x_3$-axis. In these principal axes of $\chi$, we have

$$\chi = (\rho/12)(11.3e_1e_1 + 0.708e_2e_2)$$  (4.4)

and the (non-zero) components of $\beta$ are as follows:

$$\beta_{1111} = 0.887\rho, \quad \beta_{2222} = 0.005\rho, \quad \beta_{1122} = \beta_{1212} = 0.053\rho$$  (4.5a)

$$\beta_{1112} = -0.001\rho, \quad \beta_{1222} = 0.010\rho$$  (4.5b)

The error due to neglect of the $\beta$-term, as measured by the Euclidean norm,

$$\delta = \|\Delta S^{(\beta)}\|/\|\Delta S\|$$  (4.6)

where, in accordance with (4.1), $S = S^{(0)} + \Delta S^{(x)} + \Delta S^{(\beta)}$. The error depends on crack density $\rho$ and Poisson’s ratio $v_0$. At $\rho = 0.12$ (a substantial value, at which the increase in the effective compliance $S_{1111}$ is about 75%) and $v_0 = 0.3$, the error $\delta = 0.033$, i.e. is quite small (we note that the ratio $\|\Delta S^{(\beta)}\|/\|\Delta S^{(x)}\|$ that characterizes relative contributions of the $\chi$- and $\beta$-terms, is 0.095).

Remark. Although the error $\delta = 0.033$ due to omission of the $\beta$-term, being relatively small, is not entirely negligible, the contribution of the $\beta$-term to the non-orthotropic part (4.5b) of the overall response is negligible, i.e. the main contribution of this term is orthotropic, with the orthotropy axes coinciding with the principal axes of $\chi$. This explains the conclusion of the above mentioned computational studies of Grechka and Kachanov [11] that orthotropy holds with good accuracy.

4.2. Approximate elastic symmetry of plasma-sprayed coating

We consider anisotropic elastic stiffnesses of a (free standing) plasma-sprayed coating that are approximately transversely isotropic, with the plane of isotropy normal to the spraying direction. Guided by the ultrasonic measurements of Parthasarathi et al. [20] we rephrase their experimental data by assuming that the data have been collected from a cubic specimen cut from the coating with its orientation with respect to the spraying direction given by Euler’s angles $\phi = \pi/4, \psi = \pi/6, \theta = \pi/3$ and introducing some “noise” in the data that
would be consistent with the level of data accuracy. In the coordinate system aligned with the cube, the stiffnesses are as follows:

\[
C_{ijkl} = \begin{bmatrix}
107.2 & 32.3 & 23.2 & 7.1 & 4.4 & 1.5 \\
32.3 & 116.3 & 33.2 & 6.9 & 4.9 & 5.6 \\
23.2 & 33.2 & 110.0 & 3.0 & 2.2 & 7.9 \\
7.1 & 6.9 & 3.0 & 41.6 & 1.4 & 0.0 \\
4.4 & 4.9 & 2.2 & 1.4 & 39.8 & -0.2 \\
1.5 & 5.6 & 7.9 & 0.0 & -0.2 & 42.3
\end{bmatrix} \text{ GPa (4.7)}
\]

We form matrices

\[
C_{ijjk} = \begin{bmatrix}
162.7 & 15.0 & 11.5 \\
15.0 & 181.8 & 17.0 \\
11.5 & 17.0 & 166.4
\end{bmatrix} \text{ GPa, } C_{ikhk} = \begin{bmatrix}
189.3 & 8.5 & 6.6 \\
8.5 & 200.2 & 9.7 \\
6.6 & 9.7 & 191.4
\end{bmatrix} \text{ GPa (4.8)}
\]

Solving the eigenvalue problem for them we find that the matrices of rotation from the original coordinate system to the principal axes of the two matrices are, correspondingly,

\[
Q_1 = \begin{bmatrix}
0.795 & -0.442 & 0.434 \\
-0.057 & 0.661 & 0.749 \\
-0.604 & -0.606 & 0.501
\end{bmatrix}; \quad Q_2 = \begin{bmatrix}
0.785 & -0.424 & 0.434 \\
-0.041 & 0.660 & 0.749 \\
-0.618 & -0.620 & 0.500
\end{bmatrix} \text{ (4.9)}
\]

It is seen that matrices (4.9) are close, so that their eigenvectors (given by their columns) are close. The average between the two is

\[
n_1 = \begin{bmatrix}
0.43 \\
0.75 \\
0.5
\end{bmatrix}; \quad n_{II} = \begin{bmatrix}
-0.44 \\
0.66 \\
-0.61
\end{bmatrix}; \quad n_{III} = \begin{bmatrix}
0.79 \\
-0.05 \\
-0.61
\end{bmatrix} \text{ (4.10)}
\]

In the coordinate system \(n_1, n_{II}, n_{III}\), the stiffness matrix takes the form:

\[
\tilde{C}_{ijkl} = \begin{bmatrix}
134.6 & 33.1 & 33.9 & 0.3 & 1.4 & 1.3 \\
33.1 & 99.9 & 20.2 & 2.9 & 0.6 & 0.4 \\
33.9 & 20.2 & 102.0 & -2.4 & 0.6 & 0.9 \\
0.3 & 2.9 & -2.4 & 45.3 & -0.2 & -0.5 \\
1.4 & 0.6 & 0.6 & -0.2 & 38.3 & 0.0 \\
1.3 & 0.4 & 0.9 & -0.5 & 0.0 & 38.7
\end{bmatrix} \text{ GPa (4.11)}
\]

It is obviously close to the orthotropic one (elements in the upper right 3 \(\times\) 3 corner are much smaller than the largest of the rest). The deviation from orthotropy, as measured by parameter (2.2), is \(\delta = 0.036\). Moreover, the anisotropy is close to transverse isotropy. The best transversely isotropic fit is

\[
\tilde{C}_{ijkl}^{TI} = \begin{bmatrix}
134.6 & 33.5 & 33.5 & 0.0 & 0.0 & 0.0 \\
33.5 & 103.4 & 17.7 & 0.0 & 0.0 & 0.0 \\
33.5 & 17.7 & 103.4 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 42.8 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 38.5 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 38.5
\end{bmatrix} \text{ GPa (4.12)}
\]

with the error \(\delta = 0.041\).
The best \emph{elliptically orthotropic} fit is

\[
\hat{C}^{\text{EO}}_{ijkl} = \begin{bmatrix}
136.6 & 31.3 & 31.8 & 0 & 0 & 0 \\
31.3 & 99.3 & 23.2 & 0 & 0 & 0 \\
31.8 & 23.2 & 101.6 & 0 & 0 & 0 \\
0 & 0 & 0 & 48.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 36.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 36.8 \\
\end{bmatrix} \quad \text{GPa}
\] (4.13)

with error $\delta = 0.043$ – almost the same as for the general orthotropic fit, reflecting the fact that microcracks are the main contributors to the overall compliances in such coatings. Finally, the best \emph{isotropic} fit is $\hat{C}^{\text{ISO}}_{1111} = 111.5$; $\hat{C}^{\text{ISO}}_{1212} = 41.1$ and its error is substantially larger: $\delta = 0.12$.

\subsection*{4.3. Approximate elastic symmetry of cortical bone}

Van Buskirk and Ashman [27] provided ultrasonic data on stiffnesses where only the orthotropic components were measured (the authors assumed that the non-orthotropic terms are equal to zero):

\[
C_{ijkl} = \begin{bmatrix}
20.0 & 10.91 & 11.53 & 0 & 0 & 0 \\
10.91 & 21.7 & 11.45 & 0 & 0 & 0 \\
11.53 & 11.45 & 30.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.56 & 0 & 0 \\
0 & 0 & 0 & 0 & 5.85 & 0 \\
0 & 0 & 0 & 0 & 0 & 4.74 \\
\end{bmatrix} \quad \text{GPa}
\] (4.14)

The best transversely isotropic fit of this matrix is

\[
\hat{C}^{\text{TI}}_{ijkl} = \begin{bmatrix}
20.735 & 11.025 & 11.49 & 0 & 0 & 0 \\
11.025 & 20.735 & 11.49 & 0 & 0 & 0 \\
11.49 & 11.49 & 30.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.205 & 0 & 0 \\
0 & 0 & 0 & 0 & 6.205 & 0 \\
0 & 0 & 0 & 0 & 0 & 4.855 \\
\end{bmatrix} \quad \text{Gpa}
\] (4.15)

with error $\delta = 0.030$.

The best \emph{elliptically orthotropic} fit is

\[
\hat{C}^{\text{EO}}_{ijkl} = \begin{bmatrix}
19.852 & 10.926 & 11.663 & 0 & 0 & 0 \\
10.926 & 21.482 & 11.653 & 0 & 0 & 0 \\
11.663 & 11.653 & 29.665 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.763 & 0 & 0 \\
0 & 0 & 0 & 0 & 5.983 & 0 \\
0 & 0 & 0 & 0 & 0 & 4.943 \\
\end{bmatrix} \quad \text{GPa}
\] (4.16)

with error $\delta = 0.015$.

\subsection*{4.4. Approximate elastic symmetry of carbon fiber reinforced composite}

Choy et al. [4] provided data on the elastic properties of carbon fiber reinforced composite (34% fibers) taking specimens from several locations. The data corresponding to the middle layer are as follows (the authors
measured only the orthotropic components, implicitly assuming the non-orthotropic terms to be equal to zero):

\[
C_{ijkl} = \begin{pmatrix}
15.1 & 6.3 & 6.7 & 0 & 0 & 0 \\
6.3 & 10.9 & 6.4 & 0 & 0 & 0 \\
6.7 & 6.4 & 36.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.3 & 0 & 0 \\
0 & 0 & 0 & 0 & 5.8 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.8
\end{pmatrix} \text{ GPa}
\]

(4.17)

The best transversely isotropic fit is

\[
\tilde{C}_{ijkl}^{TI} = \begin{pmatrix}
12.725 & 6.575 & 6.55 & 0 & 0 & 0 \\
6.575 & 12.725 & 6.55 & 0 & 0 & 0 \\
6.55 & 6.55 & 36.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4.55 & 0 & 0 \\
0 & 0 & 0 & 0 & 4.55 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.075
\end{pmatrix} \text{ GPa}
\]

(4.18)

with error $\delta = 0.103$.

The best *elliptically orthotropic* fit is

\[
\tilde{C}_{ijkl}^{EO} = \begin{pmatrix}
13.987 & 6.189 & 7.925 & 0 & 0 & 0 \\
6.189 & 10.245 & 7.167 & 0 & 0 & 0 \\
7.925 & 7.167 & 34.009 & 0 & 0 & 0 \\
0 & 0 & 0 & 4.067 & 0 & 0 \\
0 & 0 & 0 & 0 & 7.025 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.189
\end{pmatrix} \text{ GPa}
\]

(4.19)

with error $\delta = 0.095$. Note that the latter error is noticeable for the reason that, for stiff fibers, the elliptic orthotropy has lower accuracy than, for example, for cracks (see maps of accuracy of representations of fourth-rank compliance contribution tensor in terms of second-rank tensors given by Sevostianov and Kachanov [21]).

5. Concluding remarks

Issues related to approximate elastic symmetries are systematically examined. Tests for the presence of approximate elastic symmetries are formulated.

We also discuss the elliptic orthotropy – the special kind of orthotropy that holds, with some approximation, for a broad class of materials with inhomogeneities. In this case, the number of independent elastic constants is reduced to six (and to only four, in the case of crack-induced anisotropy), for any orientation distribution of inhomogeneities. Results are illustrated by several examples related to the effective elastic properties of heterogeneous materials.

We note that the concept of approximate symmetry is relevant for physical properties other than elasticity as well. If these properties are characterized by fourth-rank tensors, the analysis is similar to the one for elasticity. If they are characterized by a symmetric second-rank tensor $K$ (conductivity being an example) these issues are resolved easily: the orthotropic symmetry always holds, and, in the principal representation

\[
K = k_1 e_1 e_1 + k_2 e_2 e_2 + k_3 e_3 e_3
\]

the proximity to transverse isotropy or to isotropy is readily established by examining the three eigenvalues. For example, the best isotropic fit is given by the average:

\[
\{k_1, k_2, k_3\} \rightarrow k \equiv (k_1 + k_2 + k_3)/3
\]

(5.2)
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Appendix A. Tests for orthotropy and transverse isotropy

We consider a stiffness matrix $C_{ijkl}$ in certain coordinate system $x_1, x_2, x_3$ and assume that it does not have the orthotropic form in this system, so that, if the orthotropic symmetry is present, it is not obvious. We are seeking a test for orthotropy, i.e. we pose two questions:

- Does a coordinate system $\bar{x}_1, \bar{x}_2, \bar{x}_3$ exist such that the stiffness matrix $\overline{C}_{ijkl}$ has the orthotropic form in this system?
- If the answer is affirmative, what is the orientation of the system $\bar{x}_1, \bar{x}_2, \bar{x}_3$ (the principal axes of orthotropy) with respect to the original one, $x_1, x_2, x_3$?

Such a test has been established – in a more general form that includes the monoclinic symmetry – by Cowin and Mehrabadi [5]. In cases of orthotropy or transverse isotropy, their test can be rephrased in somewhat simpler terms, as follows.

**Test for orthotropy.** We observe that, if the material is orthotropic and $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are the principal axes of orthotropy, then

**A.** The matrix of the symmetric second-rank tensor $\overline{C}_{ijkk}$ is diagonal in these axes;

**B.** The matrix of the symmetric second-rank tensor $\overline{C}_{ijkj}$ is diagonal in these axes as well.

Therefore, starting with matrix $C_{ijkl}$ in the original coordinate system $x_1, x_2, x_3$, we solve the eigenvalue problem for the two second-rank tensors, $C_{ijkk}$ and $C_{ikjk}$. The necessary (but not sufficient) condition for orthotropy is that the eigenvectors of the two tensors coincide. In this case, we denote them $x_I, x_{II}, x_{III}$. Then we transform the stiffness matrix to this coordinate system and verify whether the matrix $C_{ijkl}$ in this system indeed has the orthotropic form.

**Test for transverse isotropy** consists in (1) verifying the orthotropic symmetry (as described above) and (2) verifying whether two of the eigenvalues coincide for each of the two tensors, $C_{ijkk}$ and $C_{ikjk}$.

We further add that, in the two-dimensional case, the test for orthotropy can be written in a simple explicit form, by requiring that six compliances $S_{ijkl}$ are expressed in terms of only five quantities: four non-zero components in the principal axes of orthotropy and angle $\phi$ between these axes and the originally used coordinate system. The test is given by the relation

$$
2(2S_{1122} - S_{1111} - S_{2222}) + (8 - 3\sin^2 2\phi)S_{1212}\sin 4\phi + (S_{1112} - S_{2221})(-8 + 16\sin^2 2\phi - 3\sin^4 2\phi) = 0
$$

(A.1)

and

$$
\tan 2\phi = 2\frac{S_{1112} + S_{2221}}{S_{2222} - S_{1111}}
$$

(A.2)

References


