Analytical model for coherent hemodynamics spectroscopy (CHS)

The CHS model describes sinusoidal oscillations at a given angular frequency \( \omega \). Oscillatory quantities are represented by phasors that are indicated in bold face. The model expressions for \( O(\omega) \), \( D(\omega) \), \( T(\omega) \) (i.e. the phasors that describe the oscillations of oxy-, deoxy- and total hemoglobin concentrations) as a function of \( \text{cbv}(\omega) \), \( \text{cbf}(\omega) \), and \( \text{cmro}_2(\omega) \) (i.e. the phasors that describe the oscillations of cerebral blood volume, blood flow, and metabolic rate of oxygen) are as follows:

\[
O(\omega) = cHb \left[ S^{(a)} \text{CBV}^{(a)}_0 \text{cbv}^{(a)}(\omega) + S^{(v)} \text{CBV}^{(v)}_0 \text{cbv}^{(v)}(\omega) \right] + \\
+ cHb \left[ \frac{(S^{(c)})}{S^{(v)}} \left( (S^{(c)}) - S^{(v)} \right) \text{CBV}^{(c)}_0 H^{(c)}_{RC-LP}(\omega) + (S^{(a)} - S^{(v)}) \text{CBV}^{(v)}_0 H^{(v)}_{G-LP}(\omega) \right] \left[ \text{cbf}(\omega) - \text{cmro}_2(\omega) \right]
\]

\[
D(\omega) = cHb \left[ (1 - S^{(a)}) \text{CBV}^{(a)}_0 \text{cbv}^{(a)}(\omega) + (1 - S^{(v)}) \text{CBV}^{(v)}_0 \text{cbv}^{(v)}(\omega) \right] + \\
- cHb \left[ \frac{(S^{(c)})}{S^{(v)}} \left( (S^{(c)}) - S^{(v)} \right) \text{CBV}^{(c)}_0 H^{(c)}_{RC-LP}(\omega) + (S^{(a)} - S^{(v)}) \text{CBV}^{(v)}_0 H^{(v)}_{G-LP}(\omega) \right] \left[ \text{cbf}(\omega) - \text{cmro}_2(\omega) \right]
\]

\[
T(\omega) = cHb \left[ \text{CBV}^{(a)}_0 \text{cbv}^{(a)}(\omega) + \text{CBV}^{(v)}_0 \text{cbv}^{(v)}(\omega) \right],
\]

where \( H^{(c)}_{RC-LP}(\omega) \) and \( H^{(v)}_{G-LP}(\omega) \) are the complex transfer function given by:

\[
H^{(c)}_{RC-LP}(\omega) = \frac{1}{\sqrt{1 + \left( \frac{\omega t^{(c)}}{e} \right)^2}} e^{-i \tan^{-1} \left( \frac{\omega t^{(c)}}{e} \right)}
\]

\[
H^{(v)}_{G-LP}(\omega) = e^{-\frac{\ln 2}{2} \left[ \omega \ 0.281 \left( t^{(c)} + t^{(v)} \right) \right]^2} e^{-i \omega \ 0.5 \left( t^{(c)} + t^{(v)} \right)}
\]

We have set \( \text{cbv}^{(c)}(\omega) = 0 \) because of the negligible dynamic dilation and recruitment of capillaries in brain tissue. Because of the high-pass nature of the cerebral autoregulation process
that regulates cerebral blood flow in response to blood pressure changes, we consider the following relationship between cbf and cbv:

$$
cbf(\omega) = k \mathcal{H}_{RC-HP}^{(AutoReg)}(\omega) cbv(\omega) = k \mathcal{H}_{RC-HP}^{(AutoReg)}(\omega) \left[ \frac{CBV_0^{(a)}}{CBV_0} cbv^{(a)}(\omega) + \frac{CBV_0^{(v)}}{CBV_0} cbv^{(v)}(\omega) \right],
$$

where $k$ is the inverse of the modified Grubb exponent, $\mathcal{H}_{RC-HP}^{(AutoReg)}(\omega)$ is the RC high-pass transfer function with cutoff frequency $f_c^{(AR)}$ that describes the effect of autoregulation.

**Matlab code**

The matlab code considers the following phasor ratios, which yield CHS spectra of amplitude ratios and phase differences:

$$
\frac{D(\omega)}{O(\omega)} = \frac{|D(\omega)|}{|O(\omega)|} e^{i[\text{Arg}(D(\omega)) - \text{Arg}(O(\omega))]} \\
\frac{O(\omega)}{T(\omega)} = \frac{|O(\omega)|}{|T(\omega)|} e^{i[\text{Arg}(O(\omega)) - \text{Arg}(T(\omega))}.
$$

Typical CHS spectra are as follows: