Supplemental material for: Stable and unstable development of an interfacial sliding instability

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LOCAL AND NON-LOCAL ELASTIC INTERACTIONS

We are concerned with the quasi-static acceleration of slip, in which elasticity introduces instantaneous interactions between points in space. That interaction may be local or non-local and is dependent on the elastic configuration. Here we examine a configuration in which interactions are generally non-local, except, as we show, in a particular limit where they become local: in- or anti-plane slip near a boundary. In such circumstances, we may generally write [1]

$$\mathcal{L}(V) = \frac{\mu'}{2\pi} \int_{-\infty}^{\infty} \frac{\partial V}{\partial s} \left[\frac{1}{s-x} + k_b(s-x) \right] ds \qquad \text{(S.1)}$$

where μ' is mode-dependent, the first convolution kernel within brackets is that which appears in the absence of a boundary, and the functional form of k_b depends on the nature of the boundary and is also mode-dependent.

Such an operator can be conveniently evaluated by way of a Fourier transformation. Using the convention

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx \qquad \text{(S.2)}$$

$$\mathcal{F}^{-1}[g(k)] = \int_{-\infty}^{\infty} g(k) e^{i2\pi kx} dk \qquad \text{(S.3)}$$

with wavenumber $k = 1/\lambda$ for a wavelength λ, we take advantage of the properties of the Fourier transform regarding a transform of a convolution and that of a derivative to write the transform of $\mathcal{L}$ as

$$\mathcal{F}\left[\mathcal{L}(V)\right] = -\mu' ik \mathcal{F}(V) \left[\mathcal{F}\left[\frac{1}{x}\right] + \mathcal{F}[k_b(x)] \right] \qquad \text{(S.4)}$$

where $\mathcal{F}[1/x] = -\pi i \,\mathrm{sgn}(k)$ and the transform of k_b depends on its form.

For the case of mode-II or mode-III slip between two elastic half-spaces, there is no additional kernel k_b. For in-plane (mode-II) rupture below and parallel to a free surface [2]

$$k_b(\zeta) = \frac{-\zeta}{4h^2+\zeta^2} + \frac{8h^2\zeta}{(4h^2+\zeta^2)^2} + \frac{4h^2\zeta^3 - 48h^4\zeta}{(4h^2+\zeta^2)^3} \qquad \text{(S.5)}$$

and its transform is

$$\mathcal{F}[k_b(x)] = 2\pi i \,\mathrm{sgn}(k) \left[\frac{1}{2} - 2\pi|hk| + (2\pi|hk|)^2 \right] e^{-4\pi|hk|} \qquad \text{(S.6)}$$

Performing a Taylor expansion of (S.6) about the shallow limit $|hk| = 0$,

$$\mathcal{F}\left[k_b(x)\right] \approx 2\pi i \,\mathrm{sgn}(k) \left[\frac{1}{2} - 4\pi|hk| + O(|hk|^2) \right] \qquad \text{(S.7)}$$

which when combined with (S.4), leads to

$$\mathcal{F}\left[\mathcal{L}(V)\right] \approx \frac{2\mu h}{1-\nu} (2\pi ik)^2 \mathcal{F}(V) + O(|hk|^2) \qquad \text{(S.8)}$$

Performing the inverse Fourier transform of the resulting expression, we find that, to leading order, the operator is local

$$\mathcal{L}(V) \approx \frac{2\mu h}{(1-\nu)} \frac{\partial^2 V}{\partial x^2} + O(|hk|^2) \qquad \text{(S.9)}$$

As may be inferred from the form of the continued expansion of (S.8), the higher order terms in (S.9) may be written as a sequence of higher order partial derivatives. For (S.11) with (S.5), $\mathcal{L}(V)$ returns an even (or odd) function when V is even (or odd) [3], implying that the sequence is one of even-ordered derivatives.

For anti-plane (mode-III) rupture k_b is given by the first term of (S.5) and its transform is the first term in (S.6) after carrying out the products. Performing a Taylor expansion of the transform and taking the inversion, leads to a similar result such that, in the shallow limit,

$$\mathcal{L}(V) = E'h \frac{\partial^2 V}{\partial x^2},\; E' = \begin{cases} 2\mu/(1-\nu) & \text{mode-II} \\ \mu & \text{mode-III} \end{cases} \qquad \text{(S.10)}$$

That $\mathcal{L}$ has the form of the local operator (S.10) when variations in slip occurs over lengthscales much longer than the depth h, implies that the response of the elastic halfspace reduces to that of a one-dimensional compressible column of material sliding on an effectively rigid base. Such simplified models have been extensively used for problems where variations in sliding may be comparable to or longer than the depth (e.g., in models of landslide motion [4, 5]); or as model continuum elastic systems, often in which an approximate inertial term is included, allowing for dynamic rupture [6–10], which may have novel applications elsewhere (e.g., in representing stick-slip motion of ice streams due to local basal rupture [11–14]). A series of spring block sliders is also frequently used as a model system, of which (S.10) is the continuum limit. Replacing the non-local operator with a local one simplifies analysis and computational effort in numerical solutions.

For two elastically identical half-spaces in contact, there are no changes to the interface normal stress. For mode-III slip on a plane below and parallel to a free surface, there are likewise no changes. However for mode-II slip, the normal stress does change due to slip. The rate of that change may be calculated by performing a similar convolution to (S.11),

$$\frac{\partial \sigma}{\partial t} = \frac{\mu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{\partial V}{\partial s} k_\sigma(s-x) ds \tag{S.11}$$

where [2]

$$k_\sigma(\zeta) = \frac{32h^5 - 24h^3\zeta^2}{(4h^2+\zeta^2)^3} \tag{S.12}$$

The Fourier transform of the convolution is

$$\mathcal{F}[\partial\sigma/\partial t] = -\frac{\mu}{1-\nu} ik\mathcal{F}(V)\left[-(2\pi)^3|hk|^2 \exp^{-4\pi|hk|}\right] \tag{S.13}$$

Expanding about the shallow limit, and inverting the transform, we find

$$\frac{\partial \sigma}{\partial t} \approx -\frac{\mu h^2}{1-\nu}\frac{\partial^3 V}{\partial x^3} + O(|hk|^3) \tag{S.14}$$

The first term here is of the order of the terms neglected in (S.9), which implies that the normal stress is constant to the leading order near the shallow-depth limit. Thus, we may justifiably repeat the asymptotic analysis in the manuscript for the non-local operator (7), now using the local operator (S.10) and continuing to assume a constant normal stress, under the condition that the minimum wavelength of variation is large in comparison with h (i.e., $h/L_b \ll 1$).

FIXED-POINTS AND THEIR STABILITY UNDER LOCAL ELASTICITY

We consider the case of single-mode slip below a free surface in the limit $h \ll L_b$. We look for the blowup solutions described by the fixed points to the dynamical system (4): the functions $\mathcal{W}$, $\mathcal{P}$. We must solve (5) with (S.10): i.e., a free boundary problem to determine $\mathcal{W}$ and L. We require that $\mathcal{W}(\pm L) = 0$, $\mathcal{W}'(\pm L) = 0$. The latter condition implies that a singularity in stress rate is owed only to the divergence of the slip rate.

The solution, as it depends on a/b, is found in closed form. For now, it suffices to present the solution for limit values of a/b:

$$0 < a/b \le 0.5,\ \mathcal{W}(x) = \frac{a}{b}\left[1+\cos\left(\pi\frac{x}{L}\right)\right],\ \frac{L}{L_{bh}} = \pi$$
$$a/b \to 1,\ \mathcal{W}(x) = \frac{1-(x/L)^2}{2(1-a/b)},\ \frac{L}{L_{bh}} \to \frac{1}{1-a/b} \tag{S.15}$$

where $L_{bh} = \sqrt{E'hD_c/(\sigma b)}$ is a characteristic length-scale.

The fixed points lose stability in the same manner as the configuration considered in the main text: a cascade of Hopf bifurcations. However, for the configuration considered here, the fixed points are relatively stable and the cascade commences at a value a/b much close to 1: at $a/b \approx 0.95$, compared with $a/b \approx 0.72$ for the non-local case. That a loss of fixed points stability may require a value of a/b closer to 1, becomes apparent when comparing the scaling of L to that of the critical wavelength from the stability analysis of uniform steady-state slip, λ_{cr}. Specifically, $\lambda_{cr} \sim L_{bh}/\sqrt{1-a/b}$, and $L \sim L_{bh}/(1-a/b)$, such that the ratio L/λ_{cr} grows at a slower rate as $a/b \to 1$.

OTHER DESCRIPTIONS OF STATE OR FRICTIONAL EVOLUTION

Here we consider an application of the asymptotic analysis of the main text to two instances in which frictional evolution depends on sliding rate and its history, but differs from the particular form considered in the main text. In the first instance we show that the analysis leads to similar results: that instability may be considered as a fixed-point of a dynamical system and that the governing equations of those fixed points are congruent with those of the main text. In the second instance, in which the form of frictional evolution varies more significantly, we provide a preliminary examination of past numerical results that indicate that the analysis pursued in the main text may be applicable here as well.

We first consider a recently proposed alternative functional form for state evolution [15]:

$$\frac{\partial \theta}{\partial t} = 1 - \frac{V\theta}{D_c} - \frac{c}{b}\theta\frac{\partial\tau/\partial t}{\sigma} \tag{S.16}$$

which introduces an additional dimensionless parameter c, the increase of which allows for a transition between behavior typical of that of the aging law (near $c = 0$) to that of the slip law and provides better fits for experimental friction data [16, 17].

Using this state evolution law, we perform the same change of variable from V, θ, and t to W, Φ, and s. The resulting dynamical system has fixed points for W and Φ we dub here $\mathcal{W}_N$ and $\mathcal{P}_N$. These fixed points satisfy an equation of identical form to that of the aging law (5), when written as

$$\left(1 - \frac{a}{b}\frac{1}{C}\right) + \hat{\mathcal{L}}(\mathcal{V}) = \begin{cases} 1-\mathcal{V} & \mathcal{V} \le 1 \\ 0 & \mathcal{V} \ge 1 \end{cases} \tag{S.17}$$

where $\mathcal{V} = \mathcal{W}_N/C$, $C = 1 + c(1-a/b)$, and $\hat{\mathcal{L}} = (1+c)D_c/(\sigma b)\mathcal{L}$. Consequently, we may rescale fixed-point solutions for the aging law (e.g., those in Fig. 1) to arrive to expressions for $\mathcal{W}_N$ and L_N under the Nagata law. The behavior indicated by these solutions are in agreement with numerical and alternative analytical results for instability development [16]. Furthermore, the analysis pursued here may provide a route to understand observed deviation away from the fixed-point behavior as the parameter c is increased [16, 18] in terms of loss of fixed-point stability. In this particular case however, fixed-point stability must be understood in terms of two dimensionless parameters: c and a/b.

We now briefly consider the development of slip instability under a description for frictional strength evolution whose form differs from that frequently used to describe frictional rate and state dependence (i.e., that given at the outset of the main text), but whose experimental basis remains the same and whose differences are owed to an underlying micromechanical model. Specifically, we consider a frictional constitutive relation formulated by [8], the implications of which were further studied in [9, 10]. Here we'll focus on results presented in [10], which examines the full cycle of deformation, from quasi-static forcing towards instability that leads to dynamic rupture and, subsequently, arrest. While the authors examine the linear stability of uniform steady-state sliding, we consider briefly here what may be inferred about the instability's nonlinear development. References to figures in what follows are to those in [10].

There are several points of evidence for the nonlinear stage of instability development occurring as a fixed-point attraction similar to that observed in the main text: the nucleation patch size and its development is observed to be independent of the stress state (Fig. 1c); the pre-inertial phase of slip-rate acceleration is localized in space (Fig. 1c and Supplemental Movie 2); and a likely crossover from exponential growth of slip rate with time to a divergence that scales as $D/t_f(t)$, which also coincides with their indication of the onset of nonlinear effects (Fig. 2c), where D is a characteristic slip distance in their model, similar to D_c here).

A comparison may be drawn with the results presented here. The friction law used in [10] has similar features to the one explored here, chiefly stationary aging of contacts and logarithmic weakening with slip rate for high slip rates. Additionally, the authors use the local elastic description detailed in the preceding section. With the analogous role of the slip scales D and D_c in mind, divergence in the form of $[1+\cos(x/L_{bh})]D_c/t_f(t)$ indicated in (S.15) is close to the functional fit used in Fig. 2b: $1+\cos(2\pi x/L_c)$, where $L_c = 2\pi\sqrt{E'hD/|\partial\tau_{ss}/\partial \ln V|}$. While their fit is made within a regime in which their linear stability analysis may apply, the comparison indicates that a similar spatial profile may persist into the non-linear regime. However, we caution that this is only a cursory comparison, and a more thorough inspection of the governing equations in [10] for the existence of self-similar blowup solutions and their form should be made.

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