Qualifying Exam
EE128 -- Operating Systems

1. Describe the steps needed to create a file in a modern disk filesystem. Which disk blocks are modified on disk, and for what purposes?
2. In a multi-threaded program, two threads try to execute the following statements at the exact same time:

```c
foo += 5;
foo -= 3;
```

How many different things can happen as a result of this, and how would you assure that the desired result "foo += 2" occurs rather than the others?

3. Consider the O(1) scheduler, which implements process priority by giving higher priority processes longer slices of time in which to run. The O(1) scheduler uses dynamic priority to compensate for scheduling unfairness due to I/O; it temporarily increases the priority of a process that was previously blocked waiting for I/O. Explain why this reduces scheduling unfairness, but also why it does not make O(1) scheduling completely fair.
Probability and Statistics

In this problem consider the sequence of random variables \( X_n \) for \( n = 1, 2, \ldots \) defined as follows

\[
X_n = X_{n-1} + W_n
\]

where \( X_0 \) is equally likely to be -1 or 1 and is independent of the sequence \( W_n \).

For the first two parts of this problem suppose that \( W_n \) are independent and identically distributed (IID) Gaussian random variables with mean zero and variances all 1.

1. What are the expected value and variance of \( X_2 \)?
2. What is the exact probability density function for \( X_n \)?

Now suppose that the \( W_n \) are IID random variables uniformly distributed between \( -\frac{1}{2} \) and \( +\frac{1}{2} \).

3. What is a really good approximation for the probability density function of \( X_{1200} \)?
4. What is the exact moment generating function for the conditional distribution of \( X_{1201} \) given \( X_{1200} \)?
5. A signals engineer has been charged with building a system to estimate \( X_0 \) and has decided on the rule

\[
\hat{X}_0 = \begin{cases} 
1 & X_n > 0 \\
-1 & X_n \leq 0 
\end{cases}
\]

Either determine the smallest \( n \) such that the probability of error is less than 0.1 or explain in detail why such an \( n \) does not exist.
Digital Electronics:

a. With as few **logical gates (including tri-state buffers)** and flip-flops as possible, design a bi-directional IO interface with a microprocessor with 16-bit address line, 8-bit data lines, and IO/Mem Select (0 for IO and 1 for Memory access), Write and Read Control lines. Please use two addresses for two I/O ports: one I/O Data Port (address 0x0001) and the other I/O Data Direction Port (address 0x0003) for the data port. The Data Direction Port controls the Data Port to be input or output. Assume Isolated IO (i.e. Port Mapped IO). Indicate where you should connect IO Devices to your data port.

b. A GPIO output pin is configured as an open-drain. Please design an LED output device which is lit when the output is Z.
EE107 Communication Systems – PhD Qualifying Exam 2019

Question 1. (12.5 points)

A bandpass 8-PAM signal is generated by exciting a raised-cosine shaping filter with a 50% roll-off factor and then performing double-sideband suppressed-carrier (DBS-SC) modulation on a sinusoidal carrier at frequency $f_c = 200$ kHz, as illustrated in the figure below. The bit rate is 192 Kbps.

(a) Determine the spectrum of the modulated PAM signal and sketch it. Clearly label all important frequencies.

(b) Draw the block diagram illustrating the optimum demodulator/detector for the received signal, which is equal to the transmitted signal plus additive white Gaussian noise. Clearly specify the impulse response of the filter used, if any, and the decision rule for detection. Sketch the spectrum at each point in the demodulator before sampling.

Question 2. (12.5 points)

a) Eight analog microphone signals are sampled, quantized, encoded and multiplexed into digital data. Each signal has a bandwidth of 20 kHz and is sampled at 20% higher than Nyquist frequency. The quantizer has 256 levels. Six bits of synchronization and coding are added for each multiplexed frame. Sketch a diagram of the system and compute the resulting bit rate.

b) The resulting data of part a) is transmitted over a bandpass channel of bandwidth 800 kHz using an M-ary QAM system with raised cosine spectrum. Design a system with the smallest signal constellation size. Specify the roll-off factor of the raised cosine spectrum that uses the whole transmission bandwidth.
MOSFET equations for this problem:

\[
I_D = \begin{cases} 
0 \\
\frac{k[(V_{GS} - V_T) V_{DS} - \frac{1}{2} V_D^2]}{2k(V_{GS} - V_T)^2(1 + \lambda V_D)} 
\end{cases}
\]

The MOSFETs in the circuit below are matched and have \( k_n = k_p = 1 \text{ mA/V}^2 \) and \( |V_T| = 1 \text{ volt} \). Channel length modulation is \( \lambda = 0.01 \text{ v}^1 \). The DC current source is \( I_o = 0.5 \text{ mA} \) and \( V_{DD} = V_{SS} = 10 \text{ volts} \).

**Part 1.** Find the output voltage, \( v_o \), when \( v_i = 0 \). State your assumptions clearly.

**Part 2.** Assume that \( C_\infty \) is very large. Find the midband small signal gain of the circuit, \( A_v = v_o/v_i \).

**Part 3.** Assume that \( C >> C_{gs} \) and \( C >> C_{gd} \) and estimate the upper cutoff frequency (e.g., the -3 dB corner frequency) of the circuit in terms of the Thevenin equivalent source resistance, \( R \).

**Part 4.** If the channel mobility of electrons if four times greater than holes, and the n-channel devices have \( W_n/L_n = 10 \), what is \( W_p/L_p \) ?
TOPIC 1: CIRCUIT THEORY (ES3, EE21)

Problem 1: Consider the two-terminal circuit below:

(a). Find the Thevenin equivalent circuit at the terminals. Sketch the circuit and label component values.
(b). What value of load resistance will absorb the maximum power when placed at the terminals?

Problem 2: Design a non-inverting amplifier circuit using an ideal op-amp to realize a voltage gain of 60dB. The only resistors available are R <= 100kΩ. Sketch the circuit and label the input voltage, output voltage, and component values.

Problem 3: One of the most common problems for audio signals is the 60Hz hum due to the ac power distribution. You would like to remove the 60Hz hum present in your audio signal (v_in(t)) before amplification and pass most frequencies unaltered. Design a notch filter circuit (or band-stop filter) using three components: an inductor with value L=1H, a capacitor C, and resistor R. You can choose any value for R and C that meets the filter specifications. Your notch filter should be designed to null the 60Hz component in the input voltage signal and attenuate the 2kHz component by -3dB.

(a). Sketch your notch filter circuit and label all component values.
(b). Sketch the frequency response plot, |H(w)|, for your notch filter circuit.
**Problem 4:** The circuit below consists of an ideal op-amp and NMOS transistor. The input voltage \( V_{in} > 0 \) V.

(a). Write an expression for the output voltage, \( V_{out} \), as a function of the input voltage \( (V_{in}) \), R, and NMOS device parameters.

(b). What mathematical operation does this circuit perform?
Problem 1 Basic Caches (5pts)

A memory hierarchy with a direct mapped L1 cache (with no other caches in the system) uses 1-byte words and 8-bit physical addresses. The system is running a test application accesses the following memory addresses (8-bit addresses), shown in hex: 0x12, 0x23, 0x32, 0x43, 0x52, 0x63, 0x13, 0x22, 0x33, 0x42, 0x53, 0x62

The L1 cache has the following parameters:

- Word size one byte
- Block size 2 bytes
- Cache capacity: 8 bytes
- Associativity: direct mapped
- Physical address size: 8-bits
- Hit time 1 cycle (clock rate 2 Ghz)
- Miss penalty (L1 to main memory) 200 cycles

Please provide the following:

a.) Tag, Index, and offset bits for each address
b.) Record if each access was a hit or Miss and the final contents of the cache
c.) Average memory access time (AMAT)

Problem 2 Advanced Caches (5pts)

Time to design a better memory hierarchy. Use the memory addresses sequence and the direct-mapped L1 cache configuration above. The L1 cache capacity cannot change. You want to improve the average memory access time of the cache (it just needs to be faster than the direct-mapped version, no need to find the optimal design). Here are the following options, which can be combined, and the performance costs:

- Change L1 block size (free)
- Increase associativity of the L1 cache. 2-way increases hit time to 2 cycles, 3-way to 3 cycles, 4-way 4 cycles (assume LRU replacement)
- Add an L2 cache:
  - Capacity: of 16 bytes
  - Block size: same as L1 cache (configurable)
  - Hit time and transfer cost to L1: 20 cycles (direct mapped), 25 cycles (2-way), 30 cycles (4-way) (assume LRU replacement)
  - Miss penalty to main memory 180 cycles

a.) Provide your new cache design and a paragraph justifying why you designed it in that way. (4pts)
b.) AMAT for the address sequence in problem 1. (Hint: your cache should perform better) (1pt)
Problem 3: Virtual Memory (5 pts)
A virtual memory system has the following parameters.

1. Virtual address length: 46-bit
2. Physical Memory Size: 4GB
3. Page Size: 1 MB
4. Additional valid, sharing and protection bits: 4
5. 64-entry Data TLB, fully-associative

Calculate the following:

<table>
<thead>
<tr>
<th></th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Length of offset bits in virtual address (bits)</td>
</tr>
<tr>
<td>b</td>
<td>Length of offset bits in the physical address (bits)</td>
</tr>
<tr>
<td>c</td>
<td>Length of the Virtual Page Number (bits)</td>
</tr>
<tr>
<td>d</td>
<td>Length of the Physical Page Number (bits)</td>
</tr>
<tr>
<td>e</td>
<td>Total number of Virtual Pages</td>
</tr>
<tr>
<td>f</td>
<td>Size of a page table entry (bits)</td>
</tr>
<tr>
<td>g</td>
<td>Page Table Size (bits)</td>
</tr>
<tr>
<td>h</td>
<td>Size of a single TLB entry – Tag only (bits)</td>
</tr>
<tr>
<td>i</td>
<td>Size of a single TLB entry – Data only (bits)</td>
</tr>
<tr>
<td>j</td>
<td>Total memory for the entire TLB (bits)</td>
</tr>
</tbody>
</table>
Problem 4 Pipelines. (5 points)

The following segment of code executes on a 5-stage pipelined MIPS processor. The processor includes the following stages instruction fetch (IF), instruction decode (ID), execute (EX), memory access (MEM), and write back (WB). The processor includes hazard detection (inserts stalls as nops) and support data forwarding (bypassing) that can forward data from the EX/MEM and MEM/WB pipeline registers directly to the inputs to the ALU or to the data memory. The branch is initially assumed not-taken and then resolved by comparing both operands at the end of the EX stage, assuming both operands are ready, if the branch was taken the miss-speculated instructions are squashed and the correct instruction is fetched the next cycle. Assume that all registers are loaded before the start of the code segment and that instructions issue in program order.

Note: If you were unsure how a particular part of the pipeline operates, write down any assumptions you made.

```
TOP:  LW     $t1, 0($s0)
      LW     $t2, -4 ($s0)
      ADD   $t3, $t1, $t2
      SUB   $t4, $t3, $s1
      SW    $t4, -8($s0)
```

a) Show the progress of program in a pipeline diagram like the following table. Mark all stalls! (3 pts)

```
Cycle->  1   2   3   4   5   6   7   8   9   ...

LW $t1, 0($s0)  IF  ID  EX  MEM  WB
Next Instruction  IF  ID  EX  MEM  WB
```

b.) Draw arrows in your pipeline diagram showing all instances of forwarding. (2 pts)

Problem 5 Advanced Pipelines (5pts)

Imagine that the code segment in Problem 4 is now a loop. Assume that $s0 is large (> 1000 iterations)

```
TOP:  LW     $t1, 0($s0)
      LW     $t2, -4 ($s0)
      ADD   $t3, $t1, $t2
      SUB   $t4, $t3, $s1
      SW    $t4, -8($s0)
      ADDI  $s0, $s0, -12
      BNE  $s0, $zero, #TOP
```

a.) With the additional branch instructions, how many cycles are required per loop iteration? (1 pt)

b.) Name and describe a hardware technique that could reduce stalls and reduce the number of cycles per iteration. Estimate, roughly, how many cycles per iteration are saved? (2pts)

c.) Name and describe a software technique that could reduce stalls and reduce the number of cycles per iteration. Show your new code. How many cycles per iteration are saved? (2pts)
The programming problems in this section of the exam are about the linked list data structure (singly linked). You will write a series of functions using following definition of a linked list class:

```cpp
class LinkedList
{
public:
    // Assume these are written for you
    LinkedList();    // Initializes empty list
    ~LinkedList();   // Recycles all heap storage associated with list
    int length();    // Returns number of elements in list
    bool is_empty(); // Returns true if list has no elements, false otherwise

    // You will write these --- see contracts below
    void add_to_front(int new_value);
    void add_to_back(int new_value);
    int remove_from_front();

private:
    struct Node
    {
        int value;
        Node * next;
    };

    Node * front;
};
```

**Part 1:** Write the `add_to_front` method, which updates the list so that `new_value` is the first element in the list followed by all the former elements.

```cpp
void LinkedList::add_to_front(int new_value)
```
**Part 2:** Implement “add to back”. For this problem, you should use recursion, which will make the function simpler to implement. We will give you the code for the public add_to_back function, which invokes a recursive helper function; your job is to write the recursive helper function add_to_back_rec. *Hint:* look carefully at how add_to_back_rec is called; what should happen if the list is initially empty (i.e., front is null)? You will only receive full credit for a recursive solution, but partial credit will be given for one using loops.

```c
void LinkedList::add_to_back(int new_value) {
    front = add_to_back_rec(front, new_value);
}

/*
 * Return pointer to first node in a list in which new_value
 * has been added at the end.
 */
Node * LinkedList::add_to_back_rec(Node * curnode, int new_value)
```

**Part 3:** Write the remove_from_front method. It returns the integer in the first node in the list and also deletes the first node. It may crash if the list is empty.

```c
int LinkedList::remove_from_front()
```
In the next three parts, you will write functions that use the linked list class, but are not inside it, and therefore cannot directly manipulate the nodes and pointers (nor can they access front, data, next). Instead, you should use the public methods you defined above. For example, here is a function that removes all of the elements of a list and prints them out:

```cpp
void print(LinkedList * list)
{
    while ( not list->is_empty() ) {
        int val = list->remove_from_front();
        cout << val << endl;
    }
}
```

**Part 4:** Write a function (outside the class) called reverse that takes a list and returns a new list with the same elements in reverse order. Call the methods above to manipulate the lists. Note that the original list will be consumed in the process – that’s ok.

```cpp
LinkedList * reverse(LinkedList * list)
```

**Part 5:** Write a function called append that takes two lists, A and B, and joins them into one long list. In other words, add all of the elements of B to the end of A in proper order. List B will be consumed in the process.

```cpp
void append(LinkedList * A, LinkedList * B)
```
Part 6: Write a function called shuffle that takes two linked lists, A and B, and returns a new list that contains all of the elements of A and B, in alternating order. That is, the result will contain the first element of A, then the first element of B, then the second element of A, followed by the second element of B, etc. If one list is longer than the other, just add the extra elements in order at the end. Both input lists will be consumed.

LinkedList * shuffle(LinkedList * A, LinkedList * B)

Part 7: What is the running time of your append function in Part 5? If you could implement append inside the linked list class, could you reduce that running time? Explain.
Instructions:

- Please do not turn this page until told to do so. Total time allowed for this test is 120 min.

- You should **concisely indicate your reasoning and show all relevant work** for each problem. Your score will be based on an evaluation of your understanding as reflected by what you have written for an answer.

- All work you want graded must go in the exam booklet provided. Use the extra sheets provided for scratch work only.

- **There are several sheets of formulas and properties of transforms attached with this exam, which you may use to solve the problems.**
Problem 1 [50 pts]

A signal $x_p(t)$ is obtained through impulse-train sampling of a sinusoidal signal $x(t)$ whose frequency is half the sampling frequency $\omega_s$, i.e.

$$x(t) = \cos\left(\frac{\omega_s}{2} t + \phi_0\right)$$

and

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT),$$

where $T = \frac{2\pi}{\omega_s}$.

1. Find $g(t)$ such that

$$x(t) = \cos(\phi_0) \cos\left(\frac{\omega_s}{2} t\right) + g(t)$$

2. Show that $g(nT) = 0$ for all $n = 0, \pm 1, \pm 2, \ldots$

3. Use the results of previous two parts to show that if $x_p(t)$ is applied as an input to an ideal low-pass filter with cut-off $\frac{\omega_s}{2}$, the resulting output is

$$y(t) = \cos(\phi_0) \cos\left(\frac{\omega_s}{2} t\right)$$

Alternatively, you may derive Equation (1) by analyzing the problem directly in the Fourier domain, using the properties of the Fourier transform and known Fourier transform pairs.

4. What happens when $\phi_0 = \frac{\pi}{2}$? What does this tell you about the sampling rate requirement for the Nyquist sampling theorem? In particular, precisely state the Nyquist Sampling Theorem.
Problem 1:
Problem 2 [50 pts]

Two functions $u(t)$, $v(t)$ are said to be orthonormal over an interval $(a, b)$ if

$$
\int_a^b u(t)v^*(t)dt = 0 \tag{2}
$$

Here $v^*(t)$ denotes the complex-conjugate of $v(t)$. If in addition

$$
\int_a^b |u(t)|^2dt = \int_a^b |v(t)|^2dt = 1,
$$

the functions are said to be orthonormal.

A set of functions is said to be an orthogonal set (and respectively orthonormal set) if each pair of functions in the set are orthogonal (respectively orthonormal) to each other.

1. Are the functions $\sin(m\omega_0 t)$ and $\sin(n\omega_0 t)$ orthogonal over the interval $(0, T)$, where $T = \frac{2\pi}{\omega_0}$ and $m, n$ are two integers? Are they also orthonormal?

2. Show that the functions $\phi_k(t) = e^{jk\omega_0 t}$, $k = 0, \pm 1, \pm 2, \pm 3, ...$ are orthogonal to each other over any interval of length $T = \frac{2\pi}{\omega_0}$. Are they orthonormal? If not, how can you make them orthonormal?

3. Let $\{\phi_k(t)\}, k = 0, \pm 1, \pm 2, ...$ be any set of orthonormal signals over the interval $(a, b)$. Then consider a signal of the form

$$
x(t) = \sum_k a_k\phi_k(t)
$$

for some (complex valued) constants $a_k$. Show that

$$
\int_a^b |x(t)|^2dt = \sum_k |a_k|^2
$$

4. Using the development in the parts above, state and prove the Parseval’s relation for the Continuous Time Fourier Series.
Problem 2:
Table 1: Properties of the Continuous-Time Fourier Series

\[ x(t) = \sum_{k=\infty}^{+\infty} a_k e^{j k \omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j k (2\pi/T) t} \]

\[ a_k = \frac{1}{T} \int_T x(t) e^{-j k \omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-j k (2\pi/T) t} dt \]

<table>
<thead>
<tr>
<th>Property</th>
<th>Periodic Signal</th>
<th>Fourier Series Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>{x(t), y(t)} Periodic with period T and fundamental frequency ( \omega_0 = 2\pi/T )</td>
<td>( a_k ) ( b_k )</td>
</tr>
<tr>
<td>Linearity</td>
<td>( Ax(t) + By(t) )</td>
<td>( Aa_k + Bb_k )</td>
</tr>
<tr>
<td>Time-Shifting</td>
<td>( x(t - t_0) )</td>
<td>( a_k e^{-j k \omega_0 t_0} = a_k e^{-j k (2\pi/T) t_0} )</td>
</tr>
<tr>
<td>Frequency-Shifting</td>
<td>( e^{j M \omega_0 t} = e^{j (2\pi/M) T} x(t) )</td>
<td>( a_k^{M-k} )</td>
</tr>
<tr>
<td>Conjugation</td>
<td>( x^*(t) )</td>
<td>( a_k )</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>( x(-t) )</td>
<td>( a_{-k} )</td>
</tr>
<tr>
<td>Time Scaling</td>
<td>( x(\alpha t), \alpha &gt; 0 ) (periodic with period ( T/\alpha ))</td>
<td>( T \alpha a_k b_k )</td>
</tr>
<tr>
<td>Periodic Convolution</td>
<td>( \int_T x(\tau) y(t - \tau) d\tau )</td>
<td>( \sum_{l=-\infty}^{+\infty} a_l b_{k-l} )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( x(t)y(t) )</td>
<td>( j k \omega_0 a_k = j k \frac{2\pi}{T} a_k )</td>
</tr>
<tr>
<td>Differentiation</td>
<td>( \frac{dx(t)}{dt} )</td>
<td>( \frac{1}{j k \omega_0} a_k = \left( \frac{1}{j k (2\pi/T)} \right) a_k )</td>
</tr>
<tr>
<td>Integration</td>
<td>( \int_{-\infty}^{t} x(t) dt ) (finite-valued and periodic only if ( a_0 = 0 ))</td>
<td>( \Re { a_k } ) ( \Im { a_k } )</td>
</tr>
<tr>
<td>Conjugate Symmetry</td>
<td>( x(t) ) real</td>
<td>( a_k = a_{-k}^* )</td>
</tr>
<tr>
<td>for Real Signals</td>
<td></td>
<td>( \Re { a_k } = \Re { a_{-k} } )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Im { a_k } = -\Im { a_{-k} } )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Im { a_k } = -\Im { a_{-k} } )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( a_k ) purely imaginary and odd</td>
</tr>
<tr>
<td>Real and Even Signals</td>
<td>( x(t) ) real and even</td>
<td>( a_k ) real and even</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Real and Odd Signals</td>
<td>( x(t) ) real and odd</td>
<td>( a_k ) purely imaginary and odd</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Even-Odd Decomposition of</td>
<td>( { x_e(t) = \mathcal{E}e{x(t)} } [x(t) \text{ real}] ) ( \text{Re} { a_k } ) ( j \Im { a_k } )</td>
<td></td>
</tr>
<tr>
<td>Real Signals</td>
<td>( x_o(t) = \mathcal{O}d{x(t)} ) [x(t) \text{ real}]</td>
<td></td>
</tr>
<tr>
<td>Parseval’s Relation for</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Periodic Signals</td>
<td>( \frac{1}{T} \int_T</td>
<td>x(t)</td>
</tr>
</tbody>
</table>
Table 2: Properties of the Discrete-Time Fourier Series

\[
x[n] = \sum_{k=\langle N \rangle} a_k e^{j k \omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{j k (2\pi/N) n}
\]

\[
a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j k \omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j k (2\pi/N) n}
\]

<table>
<thead>
<tr>
<th>Property</th>
<th>Periodic signal</th>
<th>Fourier series coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Periodic with period N and fundamental frequency ( \omega_0 = 2\pi/N )</td>
<td>Periodic with period N</td>
</tr>
<tr>
<td>Linearity</td>
<td>( Ax[n] + By[n] )</td>
<td>( Aa_k + Bb_k )</td>
</tr>
<tr>
<td>Time shift</td>
<td>( x[n - n_0] )</td>
<td>( a_k e^{-j k (2\pi/N) n_0} )</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>( e^{j M(2\pi/N) n} x[n] )</td>
<td>( a_k )</td>
</tr>
<tr>
<td>Conjugation</td>
<td>( x^*[n] )</td>
<td>( a^*_k )</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>( x[-n] )</td>
<td>( a_{-k} )</td>
</tr>
<tr>
<td>Time Scaling</td>
<td>( x_{(m)}[n] = \left{ \begin{array}{l} x[n/m] \quad \text{if } n \text{ is a multiple of } m \ 0 \quad \text{if } n \text{ is not a multiple of } m \end{array} \right} ) (periodic with period ( mN ))</td>
<td>( \frac{1}{m} a_k ) (viewed as periodic with period ( mN ))</td>
</tr>
<tr>
<td>Periodic Convolution</td>
<td>( \sum_{r=\langle N \rangle} x[r] y[n-r] )</td>
<td>( N a_k b_k )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( x[n] y[n] )</td>
<td>( \sum_{l=\langle N \rangle} a_l b_{k-l} )</td>
</tr>
<tr>
<td>First Difference</td>
<td>( x[n] - x[n-1] )</td>
<td>( (1 - e^{-j k (2\pi/N)}) a_k )</td>
</tr>
<tr>
<td>Running Sum</td>
<td>( \sum_{k=\langle -\infty \rangle} x[k] \left{ \begin{array}{l} \text{finite-valued and} \ \text{periodic only if } a_0 = 0 \end{array} \right} )</td>
<td>( \left( \frac{1}{1 - e^{-j k (2\pi/N)}} \right) a_k )</td>
</tr>
<tr>
<td>Conjugate Symmetry for Real Signals</td>
<td>( x[n] ) real</td>
<td>( a_k = a^*_k )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Re { a_k } = \Re { a_{-k} } )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Im { a_k } = -\Im { a_{-k} } )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \mp a_k = -\mp a_{-k} )</td>
</tr>
<tr>
<td>Real and Even Signals</td>
<td>( x[n] ) real and even</td>
<td>( a_k ) real and even</td>
</tr>
<tr>
<td>Real and Odd Signals</td>
<td>( x[n] ) real and odd</td>
<td>( a_k ) purely imaginary and odd</td>
</tr>
<tr>
<td>Even-Odd Decomposition of Real Signals</td>
<td>( x_e[n] = \mathcal{E}v{ x[n]} ) ( x_o[n] = \mathcal{O}d{ x[n]} )</td>
<td>( \Re { a_k } ) ( j \Im { a_k } )</td>
</tr>
</tbody>
</table>

Parseval’s Relation for Periodic Signals

\[
\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2
\]
Table 3: Properties of the Continuous-Time Fourier Transform

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega
\]

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt
\]

<table>
<thead>
<tr>
<th>Property</th>
<th>Aperiodic Signal</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(x(t))</td>
<td>(X(j\omega))</td>
</tr>
<tr>
<td></td>
<td>(y(t))</td>
<td>(Y(j\omega))</td>
</tr>
<tr>
<td>Linearity</td>
<td>(ax(t) + by(t))</td>
<td>(aX(j\omega) + bY(j\omega))</td>
</tr>
<tr>
<td>Time-shifting</td>
<td>(x(t - t_0))</td>
<td>(e^{-j\omega t_0}X(j\omega))</td>
</tr>
<tr>
<td>Frequency-shifting</td>
<td>(e^{j\omega_0 t}x(t))</td>
<td>(X(j(\omega - \omega_0)))</td>
</tr>
<tr>
<td>Conjugation</td>
<td>(x^*(t))</td>
<td>(X^*(-j\omega))</td>
</tr>
<tr>
<td>Time-Reversal</td>
<td>(x(-t))</td>
<td>(\frac{1}{j\omega}X(-j\omega))</td>
</tr>
<tr>
<td>Time- and Frequency-Scaling</td>
<td>(x(at))</td>
<td>(\frac{1}{</td>
</tr>
<tr>
<td>Convolution</td>
<td>(x(t) * y(t))</td>
<td>(X(j\omega)Y(j\omega))</td>
</tr>
<tr>
<td>Multiplication</td>
<td>(x(t)y(t))</td>
<td>(\frac{1}{2\pi}X(j\omega)*Y(j\omega))</td>
</tr>
<tr>
<td>Differentiation in Time</td>
<td>(\frac{d}{dt}x(t))</td>
<td>(j\omega X(j\omega))</td>
</tr>
<tr>
<td>Integration</td>
<td>(\int_{-\infty}^{t} x(t)dt)</td>
<td>(\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega))</td>
</tr>
<tr>
<td>Differentiation in Frequency</td>
<td>(tx(t))</td>
<td>(j\frac{d}{d\omega}X(j\omega))</td>
</tr>
<tr>
<td>Conjugate Symmetry for Real Signals</td>
<td>(x(t)) real</td>
<td>(X(j\omega) = X^*(-j\omega))</td>
</tr>
<tr>
<td></td>
<td>(\Re{X(j\omega)} = \Re{X(-j\omega)})</td>
<td>(\Im{X(j\omega)} = -\Im{X(-j\omega)})</td>
</tr>
<tr>
<td></td>
<td>(3m{X(j\omega)} = -3m{X(-j\omega)})</td>
<td>(</td>
</tr>
<tr>
<td>Symmetry for Real and Even Signals</td>
<td>(x(t)) real and even</td>
<td>(X(j\omega)) real and even</td>
</tr>
<tr>
<td>Symmetry for Real and Odd Signals</td>
<td>(x(t)) real and odd</td>
<td>(X(j\omega)) purely imaginary and odd</td>
</tr>
<tr>
<td>Even-Odd Decomposition for Real Signals</td>
<td>(x_e(t) = \mathcal{E}{x(t)})</td>
<td>([x(t)) real</td>
</tr>
<tr>
<td></td>
<td>(\mathcal{O}{x(t)})</td>
<td>([x(t)) real</td>
</tr>
</tbody>
</table>

Parseval’s Relation for Aperiodic Signals

\[
\int_{-\infty}^{t} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega
\]
Table 4: Basic Continuous-Time Fourier Transform Pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>Fourier transform</th>
<th>Fourier series coefficients (if periodic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=-\infty}^{+\infty} a_k e^{j\omega_0 t}$</td>
<td>$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$</td>
<td>$a_k$</td>
</tr>
<tr>
<td>$e^{j\omega_0 t}$</td>
<td>$2\pi \delta(\omega - \omega_0)$</td>
<td>$a_1 = 1$, $a_k = 0$, otherwise</td>
</tr>
<tr>
<td>$\cos \omega_0 t$</td>
<td>$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$</td>
<td>$a_1 = a_{-1} = \frac{1}{2}$, $a_k = 0$, otherwise</td>
</tr>
<tr>
<td>$\sin \omega_0 t$</td>
<td>$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$</td>
<td>$a_1 = -a_{-1} = \frac{1}{2j}$, $a_k = 0$, otherwise</td>
</tr>
<tr>
<td>$x(t) = 1$</td>
<td>$2\pi \delta(\omega)$</td>
<td>$a_0 = 1$, $a_k = 0$, $k \neq 0$ (this is the Fourier series representation for any choice of $T &gt; 0$)</td>
</tr>
<tr>
<td>Periodic square wave</td>
<td>$\left{ \begin{array}{ll} 1, &amp;</td>
<td>t</td>
</tr>
<tr>
<td>$x(t + T) = x(t)$</td>
<td>$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$</td>
<td></td>
</tr>
<tr>
<td>$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$</td>
<td>$2\pi \frac{T}{\pi} \sum_{k=-\infty}^{+\infty} \delta \left( \omega - \frac{2\pi k}{T} \right)$</td>
<td>$a_k = \frac{1}{T}$ for all $k$</td>
</tr>
<tr>
<td>$x(t) \left{ \begin{array}{ll} 1, &amp;</td>
<td>t</td>
<td>&lt; T_1 \ 0, &amp;</td>
</tr>
<tr>
<td>$\frac{\sin \omega t}{\pi t}$</td>
<td>$X(j\omega) = \left{ \begin{array}{ll} 1, &amp;</td>
<td>\omega</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{j\omega} + \pi \delta(\omega)$</td>
<td></td>
</tr>
<tr>
<td>$\delta(t - t_0)$</td>
<td>$e^{-j\omega t_0}$</td>
<td></td>
</tr>
<tr>
<td>$e^{-at}u(t), \Re{a} &gt; 0$</td>
<td>$\frac{1}{a + j\omega}$</td>
<td></td>
</tr>
<tr>
<td>$te^{-at}u(t), \Re{a} &gt; 0$</td>
<td>$\left( a + j\omega \right)^2$</td>
<td></td>
</tr>
<tr>
<td>$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t), \Re{a} &gt; 0$</td>
<td>$\frac{1}{(a + j\omega)^n}$</td>
<td></td>
</tr>
</tbody>
</table>
Table 5: Properties of the Discrete-Time Fourier Transform

\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{jn\omega} d\omega \]

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \]

<table>
<thead>
<tr>
<th>Property</th>
<th>Aperiodic Signal</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( ax[n] + by[n] )</td>
<td>Periodic with ( \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) ) by ( e^{-j\omega n} ) ( X(e^{j\omega}) ) and ( e^{-j\omega n} ) ( X(e^{j\omega}) )</td>
</tr>
<tr>
<td>Time-Shifting</td>
<td>( x[n - n_0] )</td>
<td>( e^{-j\omega_0} X(e^{j\omega}) )</td>
</tr>
<tr>
<td>Frequency-Shifting</td>
<td>( e^{j\omega_0} x[n] )</td>
<td>( X(e^{j(\omega - \omega_0)}) )</td>
</tr>
<tr>
<td>Conjugation</td>
<td>( x^*[n] )</td>
<td>( X^*(e^{-j\omega}) )</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>( x[-n] )</td>
<td>( X(e^{j\omega}) )</td>
</tr>
<tr>
<td>Time Expansions</td>
<td>( x[k][n] = \begin{cases} x[n/k], &amp; \text{if } n = \text{multiple of } k \ 0, &amp; \text{if } n \neq \text{multiple of } k \end{cases} )</td>
<td>( X(e^{j\omega}) Y(e^{j\omega}) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( x[n] * y[n] )</td>
<td>( \frac{1}{2\pi} \int_{2\pi} \int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( x[n]y[n] )</td>
<td>( (1 - e^{-j\omega})X(e^{j\omega}) )</td>
</tr>
<tr>
<td>Differentiation in Time</td>
<td>( x[n] - x[n - 1] )</td>
<td>( \sum_{k=-\infty}^{+\infty} x[k] )</td>
</tr>
<tr>
<td>Accumulation</td>
<td>( \sum_{k=-\infty}^{+\infty} x[k] )</td>
<td>( \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) )</td>
</tr>
<tr>
<td>Differentiation in Frequency</td>
<td>( nx[n] )</td>
<td>( +\pi X(e^{j\theta}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) )</td>
</tr>
<tr>
<td>Conjugate Symmetry for ( x[n] ) real</td>
<td>( X(e^{j\omega}) = X^*(e^{-j\omega}) )</td>
<td>( \Re { X(e^{j\omega}) } = \Re { X(e^{-j\omega}) } )</td>
</tr>
<tr>
<td>Real Signals</td>
<td>( \Im { X(e^{j\omega}) } = -\Im { X(e^{-j\omega}) } )</td>
<td>( \Im { X(e^{j\omega}) } = \Im { X(e^{-j\omega}) } )</td>
</tr>
<tr>
<td>Symmetry for Real, Even</td>
<td>( x[n] ) real and even</td>
<td>( X(e^{j\omega}) ) real and even</td>
</tr>
<tr>
<td>Signals</td>
<td>( x[n] ) real and odd</td>
<td>( X(e^{j\omega}) ) purely imaginary and odd</td>
</tr>
<tr>
<td>Even-odd Decomposition</td>
<td>( x_e[n] = \mathcal{E}v{x[n]} )</td>
<td>( \Re { X(e^{j\omega}) } )</td>
</tr>
<tr>
<td>Real Signals</td>
<td>( x_o[n] = \mathcal{O}d{x[n]} )</td>
<td>( j\Im { X(e^{j\omega}) } )</td>
</tr>
<tr>
<td>Parseval’s Relation for Aperiodic Signals</td>
<td>( \sum_{n=-\infty}^{+\infty}</td>
<td>x[n]</td>
</tr>
</tbody>
</table>
Table 6: Basic Discrete-Time Fourier Transform Pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>Fourier transform</th>
<th>Fourier series coefficients (if periodic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{k=(N)} a_k e^{j(2\pi/N)n} )</td>
<td>( 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta \left( \omega - \frac{2\pi k}{N} \right) )</td>
<td>( a_k )</td>
</tr>
<tr>
<td>( e^{j\omega_0 n} )</td>
<td>( 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l) )</td>
<td>( \omega_0 = \frac{2\pi m}{N}, \quad a_k = \begin{cases} 1, &amp; k = m, m \pm N, m \pm 2N, \ldots \ 0, &amp; \text{otherwise} \end{cases} )</td>
</tr>
<tr>
<td>( \cos \omega_0 n )</td>
<td>( \pi \sum_{l=-\infty}^{+\infty} { \delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l) } )</td>
<td>( \omega_0 = \frac{2\pi m}{N}, \quad a_k = \begin{cases} \frac{1}{2}, &amp; k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \ 0, &amp; \text{otherwise} \end{cases} )</td>
</tr>
<tr>
<td>( \sin \omega_0 n )</td>
<td>( \frac{\pi}{j} \sum_{l=-\infty}^{+\infty} { \delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l) } )</td>
<td>( \omega_0 = \frac{2\pi m}{N}, \quad a_k = \begin{cases} -\frac{1}{2}, &amp; k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \ 0, &amp; \text{otherwise} \end{cases} )</td>
</tr>
<tr>
<td>( x[n] = 1 )</td>
<td>( 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l) )</td>
<td>( a_k = \begin{cases} 1, &amp; k = 0, \pm N, \pm 2N, \ldots \ 0, &amp; \text{otherwise} \end{cases} )</td>
</tr>
<tr>
<td>Periodic square wave</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x[n] = \begin{cases} 1, &amp;</td>
<td>n</td>
<td>\leq N_1 \ 0, &amp; N_1 &lt;</td>
</tr>
<tr>
<td>( \sum_{k=-\infty}^{+\infty} \delta[n - kN] )</td>
<td>( \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta \left( \omega - \frac{2\pi k}{N} \right) )</td>
<td>( a_k = \frac{1}{N} ) for all ( k )</td>
</tr>
<tr>
<td>( a^n u[n], \</td>
<td>a</td>
<td>&lt; 1 )</td>
</tr>
<tr>
<td>( x[n] = \begin{cases} 1, &amp;</td>
<td>n</td>
<td>\leq N_1 \ 0, &amp;</td>
</tr>
<tr>
<td>( \frac{\sin W_n}{\sin \frac{\pi}{N} W_n} = \frac{W_n}{\sin (\frac{W_n}{N})} ) ( 0 &lt; W &lt; \pi )</td>
<td>( X(\omega) = \begin{cases} 1, &amp; 0 \leq</td>
<td>\omega</td>
</tr>
<tr>
<td>( \delta[n] )</td>
<td>( 1 )</td>
<td>—</td>
</tr>
<tr>
<td>( u[n] )</td>
<td>( \frac{1}{1 - e^{-\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k) )</td>
<td>—</td>
</tr>
<tr>
<td>( \delta[n - n_0] )</td>
<td>( e^{-j\omega n_0} )</td>
<td>—</td>
</tr>
<tr>
<td>( (n+1) a^n u[n], \</td>
<td>a</td>
<td>&lt; 1 )</td>
</tr>
<tr>
<td>( (n+r-1)! a^n u[n], \</td>
<td>a</td>
<td>&lt; 1 )</td>
</tr>
<tr>
<td>Property</td>
<td>Signal</td>
<td>Transform</td>
</tr>
<tr>
<td>--------------------------------</td>
<td>-----------------------</td>
<td>----------------</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>$X(s)$</td>
<td>$R$</td>
</tr>
<tr>
<td>$x_1(t)$</td>
<td>$X_1(s)$</td>
<td>$R_1$</td>
</tr>
<tr>
<td>$x_2(t)$</td>
<td>$X_2(s)$</td>
<td>$R_2$</td>
</tr>
<tr>
<td>Linearity</td>
<td>$ax_1(t) + bx_2(t)$</td>
<td>$aX_1(s) + bX_2(s)$</td>
</tr>
<tr>
<td>Time shifting</td>
<td>$x(t - t_0)$</td>
<td>$e^{-st_0}X(s)$</td>
</tr>
<tr>
<td>Shifting in the $s$-Domain</td>
<td>$e^{s_0}x(t)$</td>
<td>$X(s - s_0)$</td>
</tr>
<tr>
<td>Time scaling</td>
<td>$x(at)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td>Conjugation</td>
<td>$x^*(t)$</td>
<td>$X^<em>(s^</em>)$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x_1(t) \ast x_2(t)$</td>
<td>$X_1(s)X_2(s)$</td>
</tr>
<tr>
<td>Differentiation in the Time Domain</td>
<td>$\frac{d}{dt}x(t)$</td>
<td>$sX(s)$</td>
</tr>
<tr>
<td>Differentiation in the $s$-Domain</td>
<td>$-tx(t)$</td>
<td>$\frac{d}{ds}X(s)$</td>
</tr>
<tr>
<td>Integration in the Time Domain</td>
<td>$\int_{-\infty}^{t} x(\tau)d(\tau)$</td>
<td>$\frac{1}{s}X(s)$</td>
</tr>
</tbody>
</table>

**Initial- and Final Value Theorems**

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$
Table 8: Laplace Transforms of Elementary Functions

<table>
<thead>
<tr>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \delta(t) )</td>
<td>1</td>
<td>All ( s )</td>
</tr>
<tr>
<td>2. ( u(t) )</td>
<td>( \frac{1}{s} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>3. (-u(-t))</td>
<td>( \frac{1}{s} )</td>
<td>( \Re{s} &lt; 0 )</td>
</tr>
<tr>
<td>4. ( \frac{t^{n-1}}{(n-1)!} u(t) )</td>
<td>( \frac{1}{s^n} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>5. (-\frac{t^{n-1}}{(n-1)!} u(-t) )</td>
<td>( \frac{1}{s^n} )</td>
<td>( \Re{s} &lt; 0 )</td>
</tr>
<tr>
<td>6. ( e^{-\alpha t} u(t) )</td>
<td>( \frac{1}{s + \alpha} )</td>
<td>( \Re{s} &gt; -\Re{\alpha} )</td>
</tr>
<tr>
<td>7. (-e^{-\alpha t} u(-t))</td>
<td>( \frac{1}{s + \alpha} )</td>
<td>( \Re{s} &lt; -\Re{\alpha} )</td>
</tr>
<tr>
<td>8. ( \frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t) )</td>
<td>( \frac{1}{(s + \alpha)^n} )</td>
<td>( \Re{s} &gt; -\Re{\alpha} )</td>
</tr>
<tr>
<td>9. (-\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(-t) )</td>
<td>( \frac{1}{(s + \alpha)^n} )</td>
<td>( \Re{s} &lt; -\Re{\alpha} )</td>
</tr>
<tr>
<td>10. ( \delta(t - T) )</td>
<td>( e^{-sT} )</td>
<td>All ( s )</td>
</tr>
<tr>
<td>11. [ \cos \omega_0 t ] ( u(t) )</td>
<td>( \frac{s}{s^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>12. [ \sin \omega_0 t ] ( u(t) )</td>
<td>( \frac{\omega_0}{s^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>13. [ e^{-\alpha t} \cos \omega_0 t ] ( u(t) )</td>
<td>( \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; -\Re{\alpha} )</td>
</tr>
<tr>
<td>14. [ e^{-\alpha t} \sin \omega_0 t ] ( u(t) )</td>
<td>( \frac{\omega_0}{(s + \alpha)^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; -\Re{\alpha} )</td>
</tr>
<tr>
<td>15. ( u_n(t) = \frac{d^n \delta(t)}{dt^n} )</td>
<td>( s^n )</td>
<td>All ( s )</td>
</tr>
<tr>
<td>16. ( u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}} )</td>
<td>( \frac{1}{s^n} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
</tbody>
</table>
### Table 9: Properties of the $z$-Transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Sequence</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>$X(z)$</td>
<td>$R$</td>
<td></td>
</tr>
<tr>
<td>$x_1[n]$</td>
<td>$X_1(z)$</td>
<td>$R_1$</td>
<td></td>
</tr>
<tr>
<td>$x_2[n]$</td>
<td>$X_2(z)$</td>
<td>$R_2$</td>
<td></td>
</tr>
<tr>
<td><strong>Linearity</strong></td>
<td>$ax_1[n] + bx_2[n]$</td>
<td>$aX_1(z) + bX_2(z)$</td>
<td>At least the intersection of $R_1$ and $R_2$</td>
</tr>
<tr>
<td><strong>Time shifting</strong></td>
<td>$x[n - n_0]$</td>
<td>$z^{-n_0}X(z)$</td>
<td>$R$ except for the possible addition or deletion of the origin</td>
</tr>
<tr>
<td><strong>Scaling in the $z$-Domain</strong></td>
<td>$e^{j\omega_0}x[n]$</td>
<td>$X(e^{-j\omega_0}z)$</td>
<td>$R$</td>
</tr>
<tr>
<td></td>
<td>$z^n x[n]$</td>
<td>$X \left( \frac{1}{z^n} \right)$</td>
<td>$z_0 R$</td>
</tr>
<tr>
<td></td>
<td>$a^n x[n]$</td>
<td>$X(a^{-1}z)$</td>
<td>Scaled version of $R$ (i.e., $</td>
</tr>
<tr>
<td><strong>Time reversal</strong></td>
<td>$x[-n]$</td>
<td>$X(z^{-1})$</td>
<td>Inverted $R$ (i.e., $R^{-1}$ = the set of points $z^{-1}$ where $z$ is in $R$)</td>
</tr>
<tr>
<td><strong>Time expansion</strong></td>
<td>$x(k)[n] = \begin{cases} x[r], &amp; n = rk \ 0, &amp; n \neq rk \end{cases}$</td>
<td>$X(z^k)$</td>
<td>$R^{1/k}$ (i.e., the set of points $z^{1/k}$ where $z$ is in $R$)</td>
</tr>
<tr>
<td><strong>Conjugation</strong></td>
<td>$x^*[n]$</td>
<td>$X^<em>(z^</em>)$</td>
<td>$R$</td>
</tr>
<tr>
<td><strong>Convolution</strong></td>
<td>$x_1[n] * x_2[n]$</td>
<td>$X_1(z)X_2(z)$</td>
<td>At least the intersection of $R_1$ and $R_2$</td>
</tr>
<tr>
<td><strong>First difference</strong></td>
<td>$x[n] - x[n - 1]$</td>
<td>$(1 - z^{-1})X(z)$</td>
<td>At least the intersection of $R$ and $</td>
</tr>
<tr>
<td><strong>Accumulation</strong></td>
<td>$\sum_{k=-\infty}^{n} x[k]$</td>
<td>$\frac{1}{1-z^{-1}}X(z)$</td>
<td>At least the intersection of $R$ and $</td>
</tr>
<tr>
<td><strong>Differentiation in the $z$-Domain</strong></td>
<td>$nx[n]$</td>
<td>$-z\frac{dX(z)}{dz}$</td>
<td>$R$</td>
</tr>
</tbody>
</table>

**Initial Value Theorem**
If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \to \infty} X(z)$.
Table 10: Some Common z-Transform Pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\delta[n]$</td>
<td>$1$</td>
<td>All $z$</td>
</tr>
<tr>
<td>2. $u[n]$</td>
<td>$\frac{1}{1-z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>3. $u[-n-1]$</td>
<td>$\frac{1}{1-z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>4. $\delta[n-m]$</td>
<td>$z^{-m}$</td>
<td>All $z$ except $0$ (if $m &gt; 0$) or $\infty$ (if $m &lt; 0$)</td>
</tr>
<tr>
<td>5. $\alpha^n u[n]$</td>
<td>$\frac{1}{1-\alpha z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>6. $-\alpha^n u[-n-1]$</td>
<td>$\frac{1}{1-\alpha z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>7. $n\alpha^n u[n]$</td>
<td>$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>8. $-n\alpha^n u[-n-1]$</td>
<td>$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>9. $[\cos\omega_0 n]u[n]$</td>
<td>$\frac{1-</td>
<td>\cos\omega_0</td>
</tr>
<tr>
<td>10. $[\sin\omega_0 n]u[n]$</td>
<td>$\frac{</td>
<td>\sin\omega_0</td>
</tr>
<tr>
<td>11. $[r^n \cos\omega_0 n]u[n]$</td>
<td>$\frac{1-</td>
<td>r\cos\omega_0</td>
</tr>
<tr>
<td>12. $[r^n \sin\omega_0 n]u[n]$</td>
<td>$\frac{</td>
<td>r\sin\omega_0</td>
</tr>
</tbody>
</table>
Question 1: There is an equivalency between the propagation of light through a dielectric material and the propagation of a high frequency AC signal in a circuit.

Light incident on a material

![Lumped element model for a transmission line](image)

<table>
<thead>
<tr>
<th>n=1</th>
<th>n=4, $\mu_r=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho=1.0\text{cm from implanted charges} = 1.5\times10^{16}\text{cm}^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$Z=120\pi$</td>
</tr>
</tbody>
</table>

a) If 100 W of light is incident on the semiconductor material at a normal incidence (shown on the left), what percent will be reflected?

b) How thick would the semiconductor have to be for the light to attenuate by 50%? Assume the light passes though only once and none reflects on the back side.

c) What would the lumped element equivalent of the situation of the left be for a representative piece of the semiconductor? Refer to the basic figure on the right as a reminder. Specifically, what values of $Z_0$, $L$, $R$, $G$, and $C$ would give you the same results? Are Heaviside's conditions a good assumption here? Why or why not?

d) On a transmission line, a horn antenna is often added to gradually couple a signal from the impedance of free space to the impedance of the transmission line. What is the optical equivalent for the system pictured on the left?

e) If you put a thick layer of gold on the backside of the semiconductor to make it a perfect reflector bouncing the light back through the semiconductor a second time, what would the circuit equivalent be on the right side of the lumped element model?

f) What is the equivalent of stub-matching for use as an anti-reflective coating for optics?

Question 2: Give an example charge configuration that would give the following fields. Note: $r$ is the spherical radial coordinate and $s$ is the cylindrical.

\[ E \sim \frac{1}{r^2} \boldsymbol{\hat{r}}, \quad b) E \sim \frac{1}{s} \boldsymbol{\hat{s}}, \quad c) E = 0 \text{ for } s < a \text{ and } E \sim \frac{1}{s} \boldsymbol{\hat{s}} \text{ for } s \geq a \]
Semiconductor 2019 Qualifier Problem:

Note: if you do not have an equation make an attempt to derive it. Otherwise, indicate how you would solve the problem if you had the information you are missing and describe best you can what the answer should be, form and function.

1) For each of the types of diodes listed below, please draw an example band structure (include arrows showing the flow of carriers) and describe the function of the diode; including how it works and the material/structural requirements that allow it to function. Also, indicate applications where these diodes are optimally used.
   a. nn-junction diode
   b. pn-junction diode
   c. schottky (metal-semiconductor) diode
   d. tunnel diode

2) For a standard heterojunction bipolar transistor made answer the following questions:
   a. Draw the band diagram for the diode. On the drawing, label the P, N, and depletion regions; as well as, the directions of carrier flows and the barriers to this flow.
   b. As a function of gate voltage show how the band diagram changes including the direction of carrier flows.
Consider a quadratic function, \( f(x) = \frac{1}{2}(x^\top D x) \), where \( x \in \mathbb{R}^p \) and \( D \) is symmetric and positive-definite, i.e., its eigenvalues follow \( 0 < \mu = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p = L \). We have the following LTI system:

\[
x_{k+1} = x_k - \alpha \cdot \nabla f(x_k),
\]

where \( \alpha \) is a positive scalar and \( \nabla f(x_k) \) denotes the derivative of \( f \) at \( x_k \). Answer the following:

(1) Find the conditions on \( \alpha \) (as a function of \( D \)) such that \( x_k \) converges to \( x^* \) such that \( \nabla f(x^*) = 0 \).

(2) Find the value of \( \alpha \) that results in the fastest convergence.

**Hint:** The derivative of \( x^\top D x \) with respect to \( x \) is \( (D + D^\top)x \).

**PhD Qualifier Question**

EE 105–Feedback Control Systems
January 2019